DOCUMENT RESUME

ED 023 601 SE 005 412

Mathematics, 7th year, Part 2.

New York City Board of Education, Brooklyn, N.Y.

Pub Date 67

Note -261p.

Available from -New York City Board of Education, Publications Sales Office, 110 Livingston Street, Brooklyn, New York 11201 ($3.00).

EDRS Price MF -$1.00 HC Not Available from EDRS.

Descriptors -Algebra, *Arithmetic, Course Content, *Curriculum, Geometry, Grade 7, Instruction, Learning Activities, Mathematical Concepts, *Mathematics, Number Concepts, *Secondary School Mathematics, *Teaching Guides

Identifiers -New York, New York City

The materials in this bulletin consist of a series of daily lesson plans for use by teachers in presenting a modern program of seventh year mathematics. In these lesson plans are developed concepts, skills , and applications. There is an emphasis on (1) an understanding of mathematical structure, (2) growth of a number system, (3) relations and operations in a number system, (4) a development of mathematical skills based on an understanding of mathematical principles, and (5) concept of set in number and in geometry. This guide contains chapters on rational numbers (addition and subtraction), open sentences, decimals, measurement, per cent, graphs, and the set of integers. (RP)

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL OFFICE OF EDUCATION POSITION OR POLICY.

MATHEMATICS

7th YEAR

PART 2

ED023601

SE 005 412

Board of Education • City of New York

NOTE TO THE TEACHER

Mathematics 7th Year is presented in two parts of which this is the second. It contains Chapters VI through XII.

Chapters I through V were published in *Mathematics 7th Year, Part I.*

Copies of this publication may be purchased from: Board of Education of the City of New York, Publications Sales Office, 110 Livingston Street, Brooklyn, N. Y. 11201. Checks should be made payable to: Auditor, Board of Education.
Price: $3.00

CURRICULUM BULLETIN • 1966-67 SERIES • No. 3b

MATHEMATICS

7th YEAR

PART 2

BOARD OF EDUCATION OF THE CITY OF NEW YORK

Permission to reproduce this copyrighted work has been granted to the Educational Resources Information Center (ERIC) and to the organization operating under contract with the U.S. Office of Education to reproduce documents included in the ERIC system by means of microfiche only, but this right is not conferred to any users of the microfiche received from the ERIC Document Reproduction Service. Further reproduction of any part requires permission of the Board of Education of the City of New York.

BOARD OF EDUCATION

ALFRED A. GIARDINO, *President*

MRS. ROSE SHAPIRO, *Vice-President*

JOSEPH G. BARKAN
AARON BROWN
THOMAS C. BURKE
LLOYD K. GARRISON
MORRIS IUSHEWITZ
JOHN H. LOTZ
CLARENCE SENIOR

Superintendent of Schools
BERNARD E. DONOVAN

Executive Deputy Superintendent of Schools
NATHAN BROWN

Deputy Superintendents of Schools
FREDERICK W. HILL
BUSINESS AFFAIRS
THEODORE H. LANG
PERSONNEL

SEELIG L. LESTER
CURRICULUM AND INSTRUCTION

COPYRIGHT 1967
BY THE BOARD OF EDUCATION OF THE CITY OF NEW YORK

Application for permission to reprint any section of this material should be made to the Superintendent of Schools, 110 Livingston Street, Brooklyn, N.Y. 11201. Reprint of any section of this material shall carry the line, "Reprinted from (title of publication) by permission of the Board of Education of the City of New York."

"PERMISSION TO REPRODUCE THIS COPYRIGHTED
MATERIAL BY MICROFICHE ONLY HAS BEEN GRANTED
BY N.Y.C. Board of Educ
TO ERIC AND ORGANIZATIONS OPERATING UNDER
AGREEMENTS WITH THE U. S. OFFICE OF EDUCATION.
FURTHER REPRODUCTION OUTSIDE THE ERIC SYSTEM
REQUIRES PERMISSION OF THE COPYRIGHT OWNER."

FOREWORD

At a time when our society is increasingly dependent upon mathematically literate citizens and upon trained mathematical manpower, it is essential that vital and contemporary mathematics be taught in our schools.

The mathematics program set forth in this publication has developed as a result of experimentation and evaluation in classroom situations. This is Part II of _Mathematics Seventh Year_. Part I, a separate bulletin, was published during the school year 1966-1967.

This bulletin represents a cooperative effort of the Office of Junior High Schools, the Bureau of Mathematics, and the Bureau of Curriculum Development.

We wish to thank the staff members who have so generously contributed to this work.

HELENE M. LLOYD
Acting Deputy Superintendent
Office of Curriculum

July, 1967

CONTENTS

Page

Chapter VI RATIONAL NUMBERS (Addition and Subtraction) 213

Addition of rational numbers; least common denominator; properties of addition; using the distributive property; subtraction of rational numbers; ratio.

Chapter VII OPEN SENTENCES 257

Meaning of open sentence; replacement set; variable; solving an open sentence; solution set; graphing the solution set of an open sentence.

Chapter VIII RATIONAL NUMBERS: DECIMAL FORM 273

Extending decimal notation; equivalent decimals; comparing decimals; rounding; addition; subtraction; multiplication; division; renaming the dividend.

Chapter IX MEASUREMENT 307

Linear measure; approximate nature of measurement; metric system; perimeter of polygon; circumference of a circle; square measure; area of a rectangular region; area of a square region; angle measurement; vertical, complementary, supplementary angles.

Chapter X RATIONAL NUMBERS: PER CENT FORM 384

Per cent as a ratio; relating decimal numerals, fractions, and per cent forms; finding a per cent of a number; per cent problems; finding what per cent one number is of another; discount; commission; simple interest.

Chapter XI GRAPHS 415

Multiple bar; multiple line; rectangle; circle

Chapter XII THE SET OF INTEGERS 441

A need for new numbers; negative integers; positive integers; set of integers; integers and the number line; order in the set of integers; addition of integers.

Index 457

INTRODUCTION

The materials in this bulletin consist of a series of daily lesson plans for use by teachers in presenting a modern program of seventh year mathematics. In these lesson plans are developed the concepts, skills, and applications of Mathematics Seventh Year. There is an emphasis on:

an understanding of mathematical structure
growth of a number system
relations and operations in a number system
a development of mathematical skills based on an understanding of mathematical principles
concept of set in number and in geometry

This bulletin is the culmination of several years of experimentation involving the cooperative efforts of the Division of Curriculum Development and the Junior High School Division. The mathematics presented in this bulletin is based upon concepts and skills which were developed in previous grades. The eighth and ninth year mathematics courses will extend the basic ideas of Mathematics Seventh Year.

The materials in this program have been tried out over a period of years in schools in all five boroughs of the city. The materials in the lesson plans reflect the classroom tryout and continued evaluation by teachers and supervisors. They have been revised a number of times in the light of these evaluations.

ORGANIZATION OF THIS BULLETIN

The content of this bulletin is arranged in the sequence in which it is to be used. It is expected that a lesson will be presented before the next numbered lesson and that each chapter will be presented before any work in the ensuing chapter is begun. Although this may seem to be a departure from the cyclical arrangement of materials found in earlier curriculum bulletins on seventh year mathematics, a cyclical approach is in fact an integral part of each chapter. For example, understanding of the concept of a number system is developed on progressively higher levels as pupils advance from an understanding of whole numbers to rational numbers to signed numbers.

Various topics for enrichment have been included. Labeled optional, they have been placed with the topics of which they are a logical outgrowth.

SUGGESTED PROCEDURE FOR USING THIS BULLETIN

It is suggested that the following procedure be considered in using this publication:

Read the entire bulletin before making plans to teach any part of it. Read each chapter in turn to become acquainted with the content and spirit of Mathematics Seventh Year and with the relationships among the topics in the course.

Study the introductory discussion in each chapter you plan to present. Note the relationship of each lesson to bhe one preceding it and the one following it. Each lesson is organized in terms of:

1. Topic
2. Aim
3. Specific Objectives
4. Procedure
5. Practice
6. Summary Questions

Amplify the practice material suggested for each lesson with additional material from suitable textbooks.

Practice in computation and in the solution of verbal problems should not be confined to the sections in which this work appears in the bulletin, but should be interspersed among other topics in order to sustain interest and provide for continuous development and reinforcement of computational skills and of problem-solving skills.

EVALUATION

An evaluation program includes not only the checking of completed work at convenient intervals, but also continued appraisal. It is a general principle of evaluation that results are checked against objectives. The objectives of this course include concepts, principles and understandings, as well as skills.

Written tests are the most frequently used instrument for evaluation and remain the chief rating tool of the teacher. Test items should be designed to test not only recall of factual items, but also the ability of the pupil to make intelligent application of mathematical principles. Some of the writing activities which teachers may use for the purpose of evaluation include:

written tests
written homework assignments
keeping of notebooks
special reports
quizzes

To evaluate pupil understanding continually, there are a number of oral activities which teachers may use such as:

pupil explanations of approaches used in new situations
pupil justification of statements
pupil restatement of problems
pupil explanation of interrelationship of ideas
pupil discovery of patterns
oral quizzes
pupil reports

Evaluation procedures also include teacher observation of pupil's work at chalkboard and of pupil's work at seat.

Self-evaluation by pupils can be encouraged through short self-marking quizzes.

DEVELOPMENT OF THE MATHEMATICS GRADE 7 PROGRAM

During the school year 1963-1964, a revised Seventh Year Mathematics scope and sequence was developed by staff members from the Division of Curriculum Development and the Junior High School Division. This scope and sequence was the basic document for writing teams which consisted of junior high school mathematics coordinators.

Preliminary materials were prepared by these teams and were reviewed by the Junior High School Mathematics Curriculum Committee. Revisions were made on the basis of the Committee's suggestions. In September 1964, the first draft of the materials was ready and was made available to teachers who were to take part in their experimental use.

These preliminary materials were tried out on an experimental basis for the first time in selected junior high schools during the school year 1964-65. A program of evaluation of these materials was set up which included chapter by chapter evaluation reports from classroom teachers, junior high school coordinators, and supervisors of mathematics in pilot schools. The materials were then revised accordingly.

The school year 1965-1966 saw the second year of experimental use of the materials with additional schools participating. Similar evaluation procedures were followed.

Final work on Part I of this bulletin, preparing it for publication, was completed in July, 1966. The final revision of Part II was completed in July, 1967.

ACKNOWLEDGMENTS

The preparation of this bulletin was under the general direction of Helene M. Lloyd, Acting Deputy Superintendent, Irving Anker, Assistant Superintendent, Office of Junior High Schools, William H. Bristow, Assistant Superintendent, Bureau of Curriculum Development, and George Grossman, Acting Director, Bureau of Mathematics.

As Chairman of the Junior High School Mathematics Curriculum Committee, Paul Gastwirth, Principal of Edward Bleeker Junior High School, coordinated the work of various committees engaged in planning for the new seventh year mathematics program.

Frank J. Wohlfort, Acting Assistant Director, Bureau of Mathematics, worked with the coordinators and arranged for the experimental tryout of the program in the junior high schools.

Miriam S. Newman, Staff Coordinator, Bureau of Mathematics, served as project leader for the writing of the experimental materials and was the principal writer of the revised materials during the school year 1965-1966, the summer of 1966, and the school year 1966-1967.

The members of the Junior High School Curriculum Committee who participated in planning the scope and sequence and in the initial preparation of lesson plans were: Spencer J. Abbott, Florence Apperman, Samuel Bier, Charles Bechtold, Anna Chuckrow, Samuel Dreskin, Charles J. Goode, Helen Halliday, Helen Kaufman, Rose Klein, Miriam S. Newman, Alfred Okin, George Paley, Meyer Rosenspan, Benedict Rubino, Joseph Segal, Ada Sheridan, Murray Soffer, Bertha Weiss, and Frank J. Wohlfort.

Junior High School Mathematics Coordinators and teachers who prepared lesson plans for classroom tryout were: Florence Apperman, Sidney Gellman, Charles J. Goode, Meyer Klein, Morris Leist, Miriam S. Newman, Benedict Rubino, Marilyn Sacco, Murray Soffer, Bertha Weiss, and Frank J. Wohlfort.

Leonard Simon, Acting Assistant Director, Bureau of Curriculum Development, was a member of the original planning group and continued throughout to assist in the planning, coordinating, revising, and preparing of the materials for publication.

Teachers and supervisors who used the material in classroom tryout and who played a part in the evaluation and revision include:

Howard Astor, Charles Bechtold, Jack Berger, Leo Blond, Lillian Bock, Arthur Bryton, Elaine Cagen, Faustina B. Carter, Kathleen O. Carpenter, Nancy Castio, Isadore Chevat, Esther S. Cohen, James Contrada, Malcolm Cooper, Patricia Cuffe, Abraham Dacher, Bettie W. Davis, Bernard Diamond, Albert Dorlando, Neal Ehrenberg, Martin Elgarten, M.J. Etra, Joan Feigenbaum, Bennett V. Fisch, Leonard Fogel, Lenore Freedman, Joel Friedberg, Freda D. Friedman, David Galerstein, Tobias Gilbert, Roslyn S. Gold, Sheldon Gold, William Goldschlager, Manuel Gonsalves, Molly K. Greenfield, Janice Greenstein, Lillian Gross, S. Arthur Gross, Jordan Grossbandler, George Grossman, Sidney S. Guss, Gerard G. Haggerty, Doreen Hall, Marvin Halpern, Thelma Hickerson, Paula M. Hofstein, Marilyn Isaacs, Gail Isquith, Sterling J. Jacobs, Frank Jones, Jr., Richard Kalman, Robert Karasik, Frieda Karsh, Irving Klein, Anne Kolasky, Mollie Kopp, Ann R. Korn, Ruth Kranz, Ann Kurzweil, Ruth Lamb, Allan Lerner, Emanuel Levine, Naomi R. Levine, Milroy Levitt, Bertha Lieberman, George Lippold, Richard Lynch, Samuel Malkin, Frank Martine, Blanche Mechlowitz, Fay Medoff, E. Gloria Meng, Joseph Moss, Alfred Okin, David Ouziel, Elias Pekale, Ethel Perin, Ellen M. Prichard, Bertha Rhodes, Louis Ricca, Jesse Rines, Sylvia Randelstein, Gerald Rosofsky, Andrea Ross, Jules Rowen, Linda Rubenstein, Harry Ruderman, Christine Sankowski, Marvin H. Sayer, Harry Schaffer, Frances Schaier, Allan Schlifstein, Michele Schlifstein, Aaron Schmerler, Ruth Schmidt, Anne R. Schwadron, Theodore Schwartzman, Evelyn T. Serino, Benjamin Shain, Marvin Silver, Ella Simpson, Beatrice D. Skala, I. William Spivack, Alan R. Starsky, Saul Steinberg, Ruth Stokes, Ernest Stossel, Zachary Summers, Arthur Vaughan, Belle Vitale, Milton Vogelstein, Louis Waltzman, Joan Warshow, Robert Winston, Paul Zalasin.

A special acknowledgment is made to Frances Moskowitz for her assistance in preparing this manuscript for publication; to Morris Leist for preparing the diagrams.

Elena Lucchini designed the cover.

Aaron N. Slotkin, Editor, had overall responsibility for design and production.

CHAPTER VI

This chapter presents procedures for reinforcing and extending the pupil's understanding of and skill in the basic operations of addition and subtraction of rational numbers. Among the important concepts and skills included in this section are the following:

computing the sum of rational numbers expressed by fractions with the same denominator; with different denominators

computing the sum of rational numbers expressed in mixed fractional form

properties of addition of rational numbers

using the distributive property of multiplication over addition in the set of rational numbers

computing the difference of rational numbers

basic concepts of ratio

The procedure presented for reinforcing pupil understanding of addition and subtraction of rational numbers is a visual one. The use of a number line is suggested to illustrate both adding and subtracting with like fractions. The number line is also used to show the difficulty of adding two rational numbers expressed as fractions with different denominators. Pupils are guided to realize that to compute the sum, or difference, of two rational numbers expressed with different denominators, we must first express them with a common denominator. Although we may use any common denominator, it is convenient to use the least such number that will serve. The least common denominator is the least common multiple of the denominators. Background for this understanding was presented in lessons 49-50.

In earlier work, pupils have understood and used the closure, commutative, and associative properties of addition with whole numbers. They now discover that these properties hold for addition with rational numbers. They find that the distributive property of multiplication over addition also is true for rational numbers. This distributive property is useful in computing a product when one factor is a whole number and the second factor is a rational number expressed in mixed fractional form. For example, we could compute the product $4\times2\frac{1}{2}$ as follows:

$$\begin{aligned} 4\times2\tfrac{1}{2} &= 4\times(2+\tfrac{1}{2}) \\ &= (4\times2) + (4\times\tfrac{1}{2}) \\ &= \ \ 8 \ \ + \ \ 2 \\ &= 10 \end{aligned}$$

Among the basic ideas developed in connection with ratio are the following:

1. A ratio is a correspondence between the numbers of two sets of objects.
2. A ratio is an ordered pair of numbers.
3. A ratio is expressed in simplest form when its terms are whole numbers and are relatively prime (have no common factor other than 1).
4. A rate is a special kind of ratio in that it usually involves a comparison between two measurements having different units of measure, as, for example, 40 miles per hour.

The study of ratio is of importance to the pupil since in mathematics and in his daily life he encounters and utilizes ratio. In introducing basic concepts of ratio, sets are used visually to develop the understanding that a ratio is a relationship or correspondence between two numbers in a definite order. This same correspondence can be expressed in an unlimited number of ways. The pupil's previous experience with the idea of greatest common factor is utilized in finding the simplest name for a ratio.

CHAPTER VI

RATIONAL NUMBERS (Addition and Subtraction)

Lessons 61-72

Lessons 61 and 62

Topic: Addition of Rational Numbers

Aim: To learn to compute the sum of rational numbers named by fractions

Specific Objectives:

Computing sums involving fractions with the same denominator
Computing sums involving fractions with different denominators

Challenge: In Mr. Martin's family budget $\frac{1}{5}$ of the income is used for rent and $\frac{1}{4}$ is used for food.

What part of his income is used for rent and food?

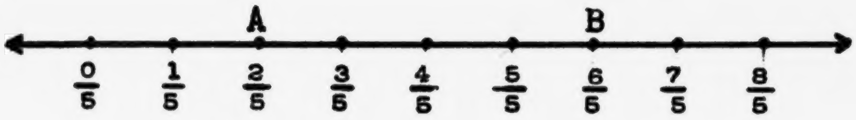
I. Procedure

A. Computing sums involving fractions with the same denominator

1. Consider $\frac{2}{5} + \frac{4}{5} = \square$.

a. What is true of both addends? (same denominator)

b. Let us picture the sum $\frac{2}{5} + \frac{4}{5}$ on the number line.



1) Since $\frac{2}{5}$ has the numerator 2, we move two spaces from 0 thereby locating point A, which corresponds to the rational number $\frac{2}{5}$. Since $\frac{4}{5}$ has the numerator 4, we move four more spaces locating point B. Point B, which represents the sum $\frac{2}{5} + \frac{4}{5}$, corresponds to the rational number $\frac{6}{5}$.

2) Thus, $\frac{2}{5} + \frac{4}{5} = \frac{6}{5}$.

c. Have pupils recall that $\frac{2}{5} = 2 \times \frac{1}{5}$ and $\frac{4}{5} = 4 \times \frac{1}{5}$.

1) $\frac{2}{5} + \frac{4}{5} = (2 \times \frac{1}{5}) + (4 \times \frac{1}{5})$

2) If we wish the distributive property to hold,

$$\begin{aligned}(2 \times \tfrac{1}{5}) + (4 \times \tfrac{1}{5}) &= (2+4) \times \tfrac{1}{5} \\ &= 6 \times \tfrac{1}{5} \\ &= \tfrac{6}{5}\end{aligned}$$

2. After several similar examples, have pupils conclude that when we add with fractions which have the same denominator, the numerator of the sum is the sum of the numerators and the denominator of the sum is the common denominator.

3. Compute the following sums.

a. $\frac{1}{3} + \frac{1}{3}$

b. $\frac{3}{7} + \frac{5}{7}$

c. $\frac{9}{4} + \frac{12}{4}$

d. $\frac{3}{10} + \frac{7}{10}$

4. Extend the procedure for adding with fractions which have the same denominator to three (or more) addends.

$$\frac{1}{8} + \frac{1}{8} + \frac{5}{8} = \frac{1 + 1 + 5}{8} = \frac{7}{8}$$

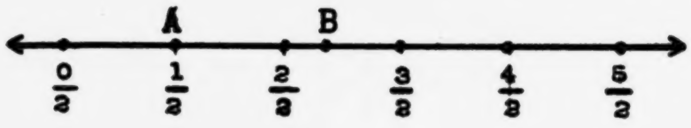
B. Computing sums involving fractions with different denominators

1. Consider $\frac{1}{2} + \frac{2}{3} = \square$

a. What is true of the denominators of the addends? (different denominators)

b. Have pupils realize the difficulty of representing this sum on the number line.

We use a number line with each unit segment partitioned into two segments of equal length (halves).



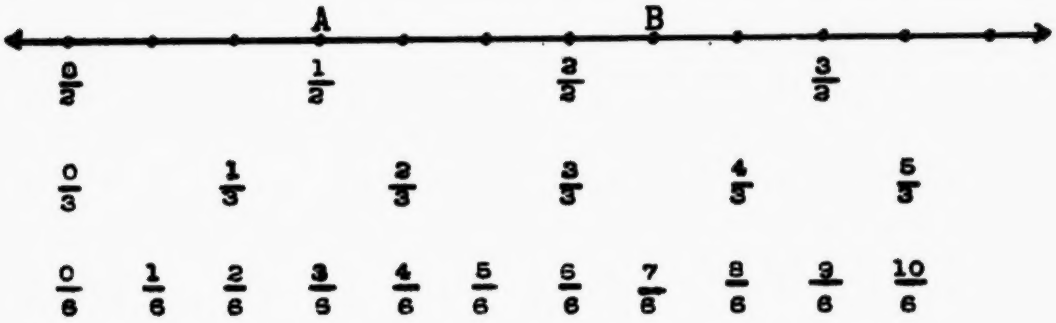
1) The point A corresponds to the rational number $\frac{1}{2}$. From point A we must move a distance equal to $\frac{2}{3}$ of a unit to the right. This will locate point B which corresponds to $\frac{1}{2} + \frac{2}{3}$. However, we cannot read the rational number which corresponds to point B.

2) Elicit that we would have a similar problem if we were to use a number line with each unit segment partitioned into 3 segments of equal length (thirds).

3) Why is it difficult to represent the sum $\frac{1}{2} + \frac{2}{3}$ on the number line pictured on page 214? (halves and thirds are different units)

c. Guide pupils to realize the need for a common unit in order to be able to represent the sum $\frac{1}{2} + \frac{2}{3}$ on a number line.

d. Elicit that a common unit may be found by first partitioning each unit segment into two segments of equal length, and then partitioning each of these segments into three segments of equal length.


A
B
$\frac{0}{2}$ $\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$
$\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$ $\frac{5}{3}$
$\frac{0}{6}$ $\frac{1}{6}$ $\frac{2}{6}$ $\frac{3}{6}$ $\frac{4}{6}$ $\frac{5}{6}$ $\frac{6}{6}$ $\frac{7}{6}$ $\frac{8}{6}$ $\frac{9}{6}$ $\frac{10}{6}$

1) Into how many line segments of equal length is each unit segment divided? (six)

Elicit that the unit segment is now divided into sixths.

2) Point A corresponds to $\frac{3}{6}$ which is another name for $\frac{1}{2}$. Since $\frac{2}{3}$ may be renamed as $\frac{4}{6}$, we then move 4 units to the right of A, locating point B. Point B which represents the sum $\frac{1}{2} + \frac{2}{3}$ corresponds to the rational number $\frac{7}{6}$.

3) Thus, $\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}$

2. Refer to challenge problem. Elicit that to solve the problem, we must compute the following sum.

$$\frac{1}{5} + \frac{1}{4} = \square$$

a. What must be true of the addends in order to compute the sum $\frac{1}{5} + \frac{1}{4}$? (they must have the same denominator)

b. How can we rename two rational numbers such as $\frac{1}{5}$ and $\frac{1}{4}$ so that they are expressed by fractions with a common denominator?

1) Consider the set of names for $\frac{1}{5}$.

$\{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \frac{5}{25}, \frac{6}{30}, \ldots\}$

The set of denominators of these names,

$\{5, 10, 15, 20, 25, 30, \ldots\}$

is the set of (non-zero) multiples of 5.

2) Consider the set names for $\frac{1}{4}$.

$\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}, \frac{6}{24}, \ldots\}$

The set of denominators of these names,

$\{4, 8, 12, 16, \ldots\}$

is the set of (non-zero) multiples of 4.

3) Elicit that the set of common denominators of $\frac{1}{4}$ and $\frac{1}{5}$ {20, 40, 60, ...} is the set of common multiples of 4 and 5.

4) What is the least common multiple of 5 and 4? (20)

What is the least common denominator of $\frac{1}{4}$ and $\frac{1}{5}$? (20)

5) Express $\frac{1}{5}$ and $\frac{1}{4}$ as fractions with the common denominator 20.

$\frac{1}{5} = \frac{1}{5} \times \frac{4}{4} = \frac{4}{20}$ $\frac{1}{4} = \frac{1}{4} \times \frac{5}{5} = \frac{5}{20}$

6) Thus, $\frac{1}{5} + \frac{1}{4} = \frac{4}{20} + \frac{5}{20} = \frac{9}{20}$

c. The part of Mr. Martin's income used for food and rent is $\frac{9}{20}$.

3. After several similar illustrations, have pupils conclude that when we add with fractions which have different denominators, we must first express them with a common denominator.

The least common denominator is the least common multiple of the denominators.

4. Pupils should realize that although it is possible to use any common denominator, it is more convenient to use the least such number that will serve.

5. Extend the procedure for adding with fractions which have different denominators to three (or more) addends.

$\frac{1}{6} + \frac{1}{8} + \frac{1}{4} = \square$

a. The denominators are 6, 8, and 4.

$6 = 2 \times 3$,

$8 = 2 \times 2 \times 2$ or 2^3

$4 = 2 \times 2$,

Therefore, the least common multiple of the denominators is $2^3 \times 3$ or 24.

What is the least common denominator?

b. $\frac{1}{6} + \frac{1}{8} + \frac{1}{4} = (\frac{1}{6} \times \frac{4}{4}) + (\frac{1}{8} \times \frac{3}{3}) + (\frac{1}{4} \times \frac{6}{6})$

$$= \frac{4}{24} + \frac{3}{24} + \frac{6}{24}$$

$$= \frac{4 + 3 + 6}{24}$$

$$= \frac{13}{24}$$

II. Practice

A. For each pair of numbers, find the least common multiple.

1. 3,5 2. 8,12 3. 6,16 4. 9,15 5. 12,18

B. Use the answers to A to obtain the least common denominators. Compute the sums. Express the sums in simplest form.

1. $\frac{1}{3} + \frac{2}{5} = \square$
2. $\frac{4}{5} + \frac{2}{3} = \square$
3. $\frac{3}{8} + \frac{5}{12} = \square$
4. $\frac{5}{6} + \frac{3}{16} = \square$
5. $\frac{4}{9} + \frac{4}{15} = \square$
6. $\frac{7}{16} + \frac{5}{6} = \square$
7. $\frac{7}{12} + \frac{11}{18} = \square$
8. $\frac{2}{15} + \frac{8}{9} = \square$
9. $\frac{7}{12} + \frac{5}{8} = \square$
10. $\frac{5}{18} + \frac{1}{12} = \square$

C. Compute the sums. Name the sums in simplest form.

1. $\frac{7}{10} + \frac{1}{10}$
2. $\frac{2}{3} + \frac{5}{6}$
3. $\frac{10}{9} + \frac{5}{6}$
4. $\frac{1}{4} + \frac{1}{3} + \frac{1}{2}$
5. $\frac{1}{4} + \frac{5}{6} + \frac{3}{8}$
6. $\frac{3}{4} + \frac{3}{10} + \frac{3}{5}$

D. The Jones family spent $\frac{1}{5}$ of its income for food, $\frac{1}{5}$ of its income for rent, and $\frac{1}{8}$ of its income for clothing.

What part of its income did the Jones family spent for food, rent, and clothing?

E. On a mathematics test, $\frac{1}{6}$ of the class had excellent scores, $\frac{1}{4}$ of the class had very good scores, and $\frac{1}{2}$ of the class had fairly good scores. The others failed.

What part of the class passed the test?

III. Summary

A. How do we compute the sum of rational numbers expressed as fractions with the same denominator?

B. How do we compute the sum of rational numbers expressed as fractions with different denominators?

C. How do we find a common denominator when adding with fractions which have different denominators?

D. Why is it an advantage to use the least common denominator when adding with fractions which have different denominators?

Lessons 63 and 64

Topic: Addition of Rational Numbers

Aim: To add rational numbers expressed in mixed fractional form

Specific Objectives:

Computing the sum of rational numbers expressed in mixed fractional form
Properties of addition of rational numbers: closure, commutativity, associativity

Challenge: The girls in Helen's club hiked $2\frac{1}{2}$ miles to Clear Lake.

The trip back, by a shorter route, was $1\frac{3}{4}$ miles.

How many miles did the girls hike on the round trip?

I. Procedure

A. Computing the sum of rational numbers expressed in mixed fractional form

1. Review changing mixed numerals to fractional form.

$$2\frac{1}{2} = 2 + \frac{1}{2} = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$$

$$1\frac{3}{4} = 1 + \frac{3}{4} = \frac{4}{4} + \frac{3}{4} = \frac{7}{4}$$

2. Refer to challenge problem.
Elicit that to solve the problem, we must compute the following sum.

$$2\frac{1}{2} + 1\frac{3}{4} = \square$$

a. We must first express the rational numbers in fractional form.

$$2\frac{1}{2} + 1\frac{3}{4} = \frac{5}{2} + \frac{7}{4}$$

The least common denominator is 4.

$$= (\frac{5}{2} \times 1) + \frac{7}{4}$$

$$= (\frac{5}{2} \times \frac{2}{2}) + \frac{7}{4}$$

Why is $\frac{2}{2}$ used as a name for 1?

$$= \frac{10}{4} + \frac{7}{4}$$

$$= \frac{17}{4} \text{ or } 4\frac{1}{4}$$

b. The girls hiked a total of $4\frac{1}{4}$ miles.

3. Have pupils practice computing sums of rational numbers expressed in mixed fractional form, first changing the mixed numerals to fractional form.

a. $1\frac{2}{3} + \frac{5}{6}$	d. $8\frac{5}{12} + 5\frac{1}{8}$
b. $3\frac{1}{7} + 2\frac{1}{3}$	e. $\frac{3}{4} + 1\frac{1}{2} + 5\frac{7}{8}$
c. $4\frac{2}{9} + 10\frac{1}{5}$	

B. Properties of addition of rational numbers

1. Closure

a. Is the sum of two rational numbers always a rational number? (Is the set of rational numbers closed under addition?)

1) Have pupils compute the sums of the following rational numbers to see whether these sums are rational numbers.

$\frac{1}{2} + \frac{1}{3}$ $\qquad \frac{7}{8} + \frac{3}{4}$ $\qquad 1\frac{1}{4} + 2\frac{3}{5}$

Is each of the sums a rational number? Why?

2) Test other cases by having the pupils suggest which rational numbers to add. Have them try to find a counter-example.

b. Why can we assume that the sum of two rational numbers is a rational number?

c. (Optional) Have pupils see that the rule for computing the sum of rational numbers expressed as fractions with the same denominator insures that the sum is a rational number.

1) The sum of the two numerators must be a whole number. (Why?)

2) The common denominator is a counting number. (Why?)

3) The sum is a rational number. (Why?)

4) Why is the sum of two rational numbers expressed as fractions with different denominators a rational number?

2. Commutativity

   a. Is addition of rational numbers commutative? For example, does $\frac{2}{3} + \frac{3}{4} = \frac{3}{4} + \frac{2}{3}$?

      1) $\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$

      2) $\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$

      3) Therefore, $\frac{2}{3} + \frac{3}{4} = \frac{3}{4} + \frac{2}{3}$

   b. After several such illustrations have pupils conclude that addition of rational numbers is commutative.

   c. (Optional) Try to establish the generalization from commutativity of addition of whole numbers.

      Thus, $\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{8 + 9}{12} = \frac{9 + 8}{12} = \frac{9}{12} + \frac{8}{12} = \frac{3}{4} + \frac{2}{3}$

3. Associativity

   a. Is addition of rational numbers associative? For example, does $(\frac{1}{2} + \frac{1}{4}) + \frac{2}{3} = \frac{1}{2} + (\frac{1}{4} + \frac{2}{3})$?

      1) $(\frac{1}{2} + \frac{1}{4}) + \frac{2}{3} = (\frac{12}{24} + \frac{6}{24}) + \frac{16}{24}$

         $= \frac{18}{24} + \frac{16}{24} = \frac{34}{24}$ or $\frac{17}{12}$

      2) $\frac{1}{2} + (\frac{1}{4} + \frac{2}{3}) = \frac{12}{24} + (\frac{6}{24} + \frac{16}{24})$

         $= \frac{12}{24} + \frac{22}{24} = \frac{34}{24}$ or $\frac{17}{12}$

      3) Therefore, $(\frac{1}{2} + \frac{1}{4}) + \frac{2}{3} = \frac{1}{2} + (\frac{1}{4} + \frac{2}{3})$.

   b. After several such illustrations, have pupils conclude that addition of rational numbers is associative.

   c. (Optional) Try to establish the generalization from associativity of addition of whole numbers.

4. Have pupils see that the use of the commutative and associative properties of addition of rational numbers makes it possible for us to compute the sum of two numbers named by mixed numerals in another way.

   a. Refer to the computation involved in the challenge problem.

   $$2\frac{1}{2} + 1\frac{3}{4} = \square$$

   We may perform this computation as follows:

   $$2\frac{1}{2} + 1\frac{3}{4} = (2 + \frac{1}{2}) + (1 + \frac{3}{4})$$

   $$= (2 + 1) + (\frac{1}{2} + \frac{3}{4}) \quad \text{Why?}$$

   $$= 3 + \frac{5}{4}$$

   $$= 3 + (1 + \frac{1}{4})$$

   $$= (3+1) + \frac{1}{4} = 4 + \frac{1}{4} \text{ or } 4\frac{1}{4}$$

   b. Elicit that in this method we add two whole numbers and two rational numbers less than 1.

5. Consider $4\frac{1}{3} = 4\frac{2}{6}$

   $$\begin{array}{rr} & 4\frac{1}{3} = 4\frac{2}{6} \\ + & \underline{3\frac{1}{2}} = \underline{3\frac{3}{6}} \\ & 7\frac{5}{6} \end{array}$$

   Which of the two methods is used in the vertical form above?

6. Compute the following sums using the vertical form.

   a. $3\frac{1}{4} + 6\frac{1}{8}$

   c. $1\frac{3}{4} + 7\frac{2}{3}$

   b. $9\frac{2}{3} + 2\frac{1}{6}$

   d. $3\frac{2}{5} + 10\frac{3}{4}$

II. Practice

A. Replace the frames.

$$2\frac{1}{4} + 7\frac{5}{6} = \left(2 + \frac{1}{4}\right) + \left(\square + \frac{5}{6}\right)$$

$$= (\triangle + 7) + \left(\frac{1}{4} + \frac{5}{6}\right)$$

$$= 9 + \frac{13}{12}$$

$$= 9 + (1 + \bigcirc)$$

$$= 10\frac{1}{12}$$

B. Compute each sum by the two methods you have learned. Which method do you think is easier?

a. $1\frac{1}{5} + 4\frac{3}{5}$

b. $15\frac{1}{2} + 36\frac{1}{3}$

C. Compute the sums.

1. $8\frac{1}{6} + 5\frac{5}{6}$

2. $14\frac{2}{5} + 6\frac{3}{10}$

3. $5\frac{7}{12} + 4\frac{2}{9}$

4. $\begin{array}{r} 6\frac{2}{7} \\ +\ 3\frac{1}{2} \\ \hline \end{array}$

5. $\begin{array}{r} 36\frac{1}{3} \\ 16\frac{5}{8} \\ +\ 6\frac{1}{4} \\ \hline \end{array}$

6. $\begin{array}{r} 5\frac{2}{3} \\ +3\frac{2}{3} \\ \hline \end{array}$

7. $\begin{array}{r} 3\frac{4}{16} \\ +1\frac{12}{16} \\ \hline \end{array}$

8. $3\frac{2}{3} + 5\frac{1}{4} + 1\frac{5}{6}$

D. Juanita used two strips of ribbon to make hair bows. The lengths of the strips were $8\frac{3}{4}$ inches and $11\frac{5}{8}$ inches. How many inches of ribbon did Juanita use in all?

E. Sam helped out in his father's store after school. He worked $1\frac{1}{4}$ hours on Monday, $2\frac{1}{2}$ hours on Tuesday, $1\frac{3}{4}$ hours on Wednesday and 2 hours on Thursday. How many hours did he work during the four days?

F. Use the commutative or associative property to replace each frame so that a true statement results.

1. $\frac{1}{8} + \square = \frac{5}{8} + \frac{1}{8}$
2. $3\frac{1}{3} + 5 = 5 + \square$
3. $(6 + \square) + 2\frac{1}{2} = 6 + (\frac{3}{4} + 2\frac{1}{2})$
4. $\frac{9}{5} + (\frac{2}{3} + 3) = (\frac{9}{5} + \frac{2}{3}) + \square$
5. $2\frac{1}{2} + \square = \square + 2\frac{1}{2}$

G. Decide whether it is easier to compute the sum as given at the left of the equals sign, or as given at the right. Then find the sum. Name the sum in simplest form.

1. $(\frac{1}{4} + \frac{2}{5}) + \frac{3}{5} = \frac{1}{4} + (\frac{2}{5} + \frac{3}{5})$
2. $(\frac{9}{8} + \frac{7}{8}) + \frac{1}{2} = \frac{9}{8} + (\frac{7}{8} + \frac{1}{2})$
3. $(\frac{5}{6} + \frac{1}{4}) + \frac{1}{4} = \frac{5}{6} + (\frac{1}{4} + \frac{1}{4})$
4. $(\frac{2}{3} + \frac{8}{9}) + \frac{1}{9} = \frac{2}{3} + (\frac{8}{9} + \frac{1}{9})$

III. Summary

A. Describe two methods of computing the sum of rational numbers expressed in mixed fractional form.

B. Name some properties of addition of whole numbers which hold for addition of rational numbers.

C. Give an illustration of how the use of the associative property of addition would help you simplify addition of rational numbers.

Lesson 65 (Optional)

Topic: Rational Numbers

Aim: To demonstrate an operation which is commutative but not associative

Specific Objectives:

To review the commutative and the associative properties of addition; of multiplication
A commutative operation need not be associative

Challenge: Is there an operation which is commutative but not associative?

I. Procedure

A. Review the commutative and the associative properties of addition; of multiplication

1. How many numbers can we add or multiply at one time? (two) Review that any operation involving two numbers is called a binary operation.

2. What property enables us to interchange the two numbers without changing the sum or product? (commutative)

3. Illustrate the commutative property of addition and the commutative property of multiplication. For example,

$$3 + 2 = 2 + 3$$
$$3 \times 2 = 2 \times 3$$

4. What property do we use to change the way we group three or more addends or factors? (associative)

5. Illustrate the associative property of addition and the associative property of multiplication. For example,

$$(2 + 3) + 5 - 2 + (3 + 5)$$
$$(2 \times 3) \times 5 = 2 \times (3 \times 5)$$

B. A commutative operation need not be associative

1. Is there an operation which is commutative but not associative?

After discussing the four fundamental operations of addition, subtraction, multiplication, division, and any others suggested by the pupils, suggest the mathematical operation of taking the average of two numbers.

2. Review the meaning of the average of two numbers.

   a. What is the average of 4 and 6? ($\frac{4+6}{2} = 5$)

   b. Let us call * (star) the operation of "taking the average of two numbers."

   c. What is 4 * 6? ($\frac{4+6}{2}$ or 5)

   d. Is * a commutative operation? (Yes)

3. Is * an associative operation?

   Let's test 4 * 6 * 8. Does 4 * (6 * 8) = (4 * 6) * 8?

   Note: 4 * 6 * 8 does not mean $\frac{4+6+8}{3}$

   a. $$\begin{aligned} 4 * (6 * 8) &= 4 * \left(\frac{6+8}{2}\right) \\ &= 4 * 7 \\ &= \frac{4+7}{2} \\ &= 5\frac{1}{2} \end{aligned}$$

   b. $$\begin{aligned} (4 * 6) * 8 &= \left(\frac{4+6}{2}\right) * 8 \\ &= 5 * 8 \\ &= \frac{5+8}{2} \\ &= 6\frac{1}{2} \end{aligned}$$

   c. $4 * (6 * 8) \neq (4 * 6) * 8$

4. Have pupils conclude that * is not an associative operation but it is a commutative operation.

5. What operation does * represent?

II. Practice

A. Find the average of $3\frac{1}{2}$ and $4\frac{2}{3}$.

B. Find the average of $4\frac{2}{3}$ and $6\frac{3}{4}$.

C. Test $3\frac{1}{2} * 4\frac{2}{3} * 6\frac{3}{4}$ for associativity.

III. Summary

A. What is the commutative law of addition? of multiplication?

B. With how many numbers are we operating when we add? when we multiply?

C. What principle enables us to add or to multiply three or more numbers?

D. What is the associative property of addition? of multiplication?

E. What operation did we discuss today which is commutative but not associative? Illustrate.

Lesson 66

Topic: Rational Numbers

Aim: To use the distributive property of multiplication over addition in the set of rational numbers

Specific Objectives:

To review the distributive property of multiplication over addition for the set of whole numbers
To use this distributive property in the set of rational numbers

Motivation: We have seen how the distributive property of multiplication over addition often helps us simplify computation with whole numbers.

Can this property help us to simplify computation with rational numbers?

I. Procedure

A. Reviewing distributive property of multiplication over addition for whole numbers

1. Which property of whole numbers is illustrated by the following?

$$\begin{aligned} 3 \times 35 &= (3 \times 30) + (3 \times 5) \\ &= 90 + 15 \\ &= 105 \end{aligned}$$

2. Replace the frame in each of the following with a numeral which will produce a true statement illustrating the distributive property of multiplication over addition.

a. $7 \times (4 + \square) = (7 \times 4) + (7 \times 5)$

b. $(\triangle \times 9) + (\square \times 9) = (4 + 6) \times 9$

B. Using the distributive property with rational numbers

1. Compute the following products.

a. $2 \times (8 + \frac{3}{4}) = \square$ b. $(2 \times 8) + (2 \times \frac{3}{4}) = \square$

$$\begin{aligned} 2 \times (8 + \tfrac{3}{4}) &= 2 \times 8\tfrac{3}{4} \\ &= 2 \times \frac{35}{4} \\ &= \frac{2 \times 35}{4} \\ &= \frac{2}{2} \times \frac{35}{2} \\ &= 17\tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} (2 \times 8) + (2 \times \tfrac{3}{4}) &= 16 + 1\tfrac{1}{2} \\ &= 17\tfrac{1}{2} \end{aligned}$$

2. **Elicit that therefore $2 \times (8 + \frac{3}{4}) = (2 \times 8) + (2 \times \frac{3}{4})$.**

   This is an illustration of the Distributive Property of Multiplication over Addition for Rational Numbers.

3. **How can we use the distributive property to simplify computation?**

   a. **Consider $5 \times 2\frac{1}{8} = \square$**

Horizontal Form	**Vertical Form**
$\begin{aligned} 5 \times 2\tfrac{1}{8} &= (5 \times 2) + (5 \times \tfrac{1}{8}) \\ &= 10 + \tfrac{5}{8} \\ &= 10\tfrac{5}{8} \end{aligned}$	$\begin{array}{rl} 2\frac{1}{8} & \\ \underline{\times\ 5} & \\ \frac{5}{8} & (5 \times \frac{1}{8}) \\ \underline{10} & (5 \times 2) \\ 10\frac{5}{8} & \end{array}$

   b. **Do we obtain the same product as in a if we compute in this way?**

   $$\begin{aligned} 5 \times 2\tfrac{1}{8} &= 5 \times \frac{17}{8} \\ &= \frac{5 \times 17}{8} \\ &= \frac{85}{8} \\ &= 10\tfrac{5}{8} \end{aligned}$$

   Which method do you prefer?

c. Consider $(\frac{3}{4} \times 45) + (\frac{3}{4} \times 55) = \square$

1) $(\frac{3}{4} \times 45) + (\frac{3}{4} \times 55) = \frac{3}{4} \times (45 + 55)$

$$= \frac{3}{4} \times 100$$

$$= 75$$

2) In which step did we use the distributive property?

3) Did the use of the distributive property simplify our work? Explain.

4. Have pupils practice using the distributive property in computing the following products.

a. $2 \times 3\frac{1}{5}$ b. $7 \times 8\frac{3}{7}$ c. $4 \times 3\frac{1}{2}$ d. $3 \times 2\frac{1}{9}$

5. Consider $6\frac{3}{4} \times 4\frac{1}{3}$.

Note to teacher: Whereas the following computation illustrates the use of the distributive property of multiplication over addition in the set of rational numbers, it is generally advisable in examples of this type to compute the product by expressing the rational numbers in fractional form before applying the rule for multiplication.

Thus, $6\frac{3}{4} \times 4\frac{1}{3}$ should be expressed as $\frac{27}{4} \times \frac{13}{3}$, and so on.

Using the distributive property, however,

$$6\frac{3}{4} \times 4\frac{1}{3} = (6 + \frac{3}{4}) \times (4 + \frac{1}{3})$$

$$= [(6 + \frac{3}{4}) \times 4] + [(6 + \frac{3}{4}) \times \frac{1}{3}]$$

$$= (6 \times 4) + (\frac{3}{4} \times 4) + (6 \times \frac{1}{3}) + (\frac{3}{4} \times \frac{1}{3})$$

$$= 24 + 3 + 2 + \frac{1}{4}$$

$$= 29\frac{1}{4}$$

II. Practice

A. Using the distributive property, write another name for each of the following:

1. $8 \times 15\frac{3}{4} = (\square \times \triangle) + (\square \times \bigcirc)$

2. $\square \times \triangle = (21 \times 7) + (21 \times \frac{2}{3})$

B. Use the distributive property to compute the products.

1. $4 \times 2\frac{1}{2}$
2. $8 \times 2\frac{3}{4}$
3. $12 \times 7\frac{2}{3}$
4. $7 \times 5\frac{1}{2}$
5. $11 \times 3\frac{1}{5}$
6. $3\frac{1}{3} \times 15$

C. Use the distributive property to simplify computation in each case.

1. $(\frac{1}{2} \times 27) + (\frac{1}{2} \times 3)$

$$\begin{aligned}(\tfrac{1}{2} \times 27) + (\tfrac{1}{2} \times 3) &= \tfrac{1}{2} \times (27 + 3)\\ &= \tfrac{1}{2} \times 30\\ &= 15\end{aligned}$$

2. $(\frac{3}{4} \times 18) + (\frac{3}{4} \times 2)$

3. $(87 \times \frac{1}{10}) + (13 \times \frac{1}{10})$

4. $(38 \times \frac{5}{6}) + (22 \times \frac{5}{6})$

5. $(103 \times \frac{9}{10}) + (97 \times \frac{9}{10})$

D. Solve each of the following in two ways.

Example $4\frac{1}{2} \times 2\frac{2}{3}$

Method 1. Using the distributive property

$$\begin{aligned}4\tfrac{1}{2} \times 2\tfrac{2}{3} &= (4 + \tfrac{1}{2}) \times (2 + \tfrac{2}{3})\\ &= [(4 + \tfrac{1}{2}) \times 2] + [(4 + \tfrac{1}{2}) \times \tfrac{2}{3}]\\ &= (4 \times 2) + (\tfrac{1}{2} \times 2) + (4 \times \tfrac{2}{3}) + (\tfrac{1}{2} \times \tfrac{2}{3})\\ &= 8 + 1 + \tfrac{8}{3} + \tfrac{1}{3}\\ &= 9 + \tfrac{9}{3}\\ &= 12\end{aligned}$$

Method 2. Expressing rational numbers in fractional form

$$4\frac{1}{2} \times 2\frac{2}{3} = \frac{9}{2} \times \frac{8}{3}$$

$$= \frac{9 \times 8}{2 \times 3}$$

$$= \frac{3 \times 3 \times 2 \times 4}{2 \times 3}$$

$$= \frac{2 \times 3}{2 \times 3} \times 3 \times 4$$

$$= 12$$

Which method did you find easier to use?

1. $6\frac{2}{3} \times 2\frac{1}{2}$

2. $3\frac{1}{3} \times 4\frac{2}{5}$

E. Ira's mother asked him to cut 5 pieces of board to be used as book shelves. Each piece is to be $2\frac{3}{4}$ feet long. He has a piece of board 15 feet long. Is this piece of board long enough for him to use in making the shelves?

F. Mary bought $1\frac{1}{4}$ yards of blue ribbon at 35¢ a yard and $2\frac{3}{4}$ yards of pink ribbon at the same price. What was the cost of all the ribbon she bought?

III. Summary

A. Name some properties of operations with whole numbers which hold for these operations with rational numbers.

B. Give an illustration of how the use of the distributive property of multiplication over addition would help you to simplify computations with rational numbers.

C. In adding rational numbers, how does the distributive property help to explain why we add numerators when we have the same denominator?

Lesson 67

Topic: Subtraction of Rational Numbers

Aim: To compute the difference of rational numbers named by fractions

Specific Objectives:

- Computing the difference of rational numbers expressed with the same denominator
- Computing the difference of rational numbers expressed with different denominators

Challenge: Dolores is planning to visit her cousin who lives in another city.

By train, the trip can be made in $2\frac{3}{4}$ hours.

By bus, the same trip takes $4\frac{1}{2}$ hours.

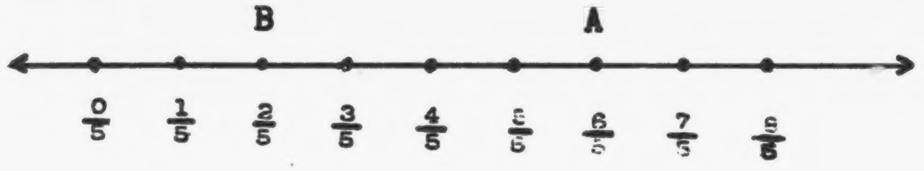
How much time would she save by taking the train?

I. Procedure

A. Computing the difference of rational numbers expressed with the same denominator

1. Consider $\frac{6}{5} - \frac{4}{5} = \square$

a. Let us first picture the difference $\frac{6}{5} - \frac{4}{5}$ on the number line. We use a number line with each unit segment partitioned into 5 segments of equal length.



1) On the number line, we locate point A which corresponds to the rational number $\frac{6}{5}$.

From A, we move back (to the left) four spaces $(\frac{4}{5})$ to locate point B.

Point B, which represents the difference $\frac{6}{5} - \frac{4}{5}$, corresponds to the rational number $\frac{2}{5}$.

2) Thus, $\frac{6}{5} - \frac{4}{5} = \frac{2}{5}$.

b. Elicit that $\frac{6}{5} = 6 \times \frac{1}{5}$ and $\frac{4}{5} = 4 \times \frac{1}{5}$.

1) $\frac{6}{5} - \frac{4}{5} = (6 \times \frac{1}{5}) - (4 \times \frac{1}{5})$

2) Using the distributive property, we have

$$(6 \times \frac{1}{5}) - (4 \times \frac{1}{5}) = (6 - 4) \times \frac{1}{5} = 2 \times \frac{1}{5} \text{ or } \frac{2}{5}$$

2. After several similar examples, have pupils conclude that if two rational numbers are expressed with the same denominator, the numerator of the difference is the difference of the numerators, and the denominator of the difference is the common denominator.

3. Compute the following differences.

a. $\frac{8}{3} - \frac{3}{3}$ b. $\frac{5}{7} - \frac{2}{7}$ c. $\frac{18}{11} - \frac{10}{11}$ d. $\frac{25}{27} - \frac{2}{27}$ e. $\frac{4}{5} - \frac{4}{5}$

B. Computing the difference of rational numbers expressed with different denominators

1. Consider $\frac{4}{5} - \frac{1}{3} = \square$

a. Have pupils recall how they compute sums of rational numbers expressed with different denominators. The numbers are renamed so that the denominators are the same.

b. Elicit that we can use a similar procedure for computing differences.

1) Rename $\frac{4}{5}$ and $\frac{1}{3}$ using the least common denominator, 15.

$\frac{4}{5} = \frac{4}{5} \times \frac{3}{3} = \frac{12}{15}$ $\frac{1}{3} = \frac{1}{3} \times \frac{5}{5} = \frac{5}{15}$

2) $\frac{4}{5} - \frac{1}{3} = \frac{12}{15} - \frac{5}{15} = \frac{7}{15}$

2. Compute the following differences.

a. $\frac{1}{2} - \frac{1}{4}$ c. $\frac{2}{3} - \frac{1}{2}$

b. $\frac{5}{12} - \frac{1}{6}$ d. $\frac{3}{4} - \frac{5}{9}$

3. Consider $2\frac{5}{9} - 1\frac{1}{6} = \square$

   a. Express $2\frac{5}{9}$ and $1\frac{1}{6}$ in fractional form. $2\frac{5}{9} - 1\frac{1}{6} = \frac{23}{9} - \frac{7}{6}$

   b. Since the denominator $9 = 3 \times 3$ or 3^2, and the denominator $6 = 3 \times 2$, the least common multiple of the denominators (the least common denominator) is $3^2 \times 2$ or 18.

   Then, $\frac{23}{9} - \frac{7}{6} = (\frac{23}{9} \times \frac{2}{2}) - (\frac{7}{6} \times \frac{3}{3})$

   $= \frac{46}{18} - \frac{21}{18}$

   $= \frac{25}{18}$ or $1\frac{7}{18}$

   c. Thus, $2\frac{5}{9} - 1\frac{1}{6} = 1\frac{7}{18}$

4. Refer to the challenge problem.
   Elicit that to solve this problem, we must compute the following difference.

   $4\frac{1}{2} - 2\frac{3}{4} = \square$

   a. $4\frac{1}{2} - 2\frac{3}{4} = \frac{9}{2} - \frac{11}{4}$ The least common denominator is 4.

   $= (\frac{9}{2} \times \frac{2}{2}) - \frac{11}{4}$

   $= \frac{18}{4} - \frac{11}{4}$

   $= \frac{7}{4}$ or $1\frac{3}{4}$

   b. Dolores would save $1\frac{3}{4}$ hours if she takes the train.

5. Have pupils also understand the following method of computing the difference $4\frac{1}{2} - 2\frac{3}{4}$, by regrouping.

   a. $4\frac{1}{2} = 4\frac{2}{4}$

   $\underline{-2\frac{3}{4}} = \underline{2\frac{3}{4}}$

   b. Since we cannot subtract $\frac{3}{4}$ from $\frac{2}{4}$, we will rename $4\frac{2}{4}$.

   We think of 1 as $\frac{4}{4}$ and we regroup $4\frac{2}{4}$ as $3\frac{6}{4}$.

$$\begin{array}{rcr} \text{Then } 4\frac{1}{2} & = & 3\frac{6}{4} \\ -\ 2\frac{3}{4} & = & 2\frac{3}{4} \\ \hline & & 1\frac{3}{4} \end{array}$$

II. Practice

A. Compute the differences.

1. $\frac{6}{11} - \frac{4}{11}$
2. $\frac{14}{25} - \frac{7}{25}$
3. $\frac{2}{3} - \frac{1}{6}$
4. $\frac{3}{4} - \frac{2}{3}$
5. $7\frac{3}{4} - 5\frac{1}{4}$
6. $15 - 6\frac{4}{7}$
7. $12\frac{2}{3} - 6\frac{1}{2}$
8. $15\frac{1}{5} - 9\frac{3}{10}$
9. $20\frac{5}{12} - 6\frac{3}{8}$
10. $126\frac{2}{7} = 59\frac{2}{3}$

B. Compute the following:

1. $\frac{2}{3} \times \left(\frac{5}{6} - \frac{1}{6}\right)$
2. $\left(\frac{2}{3} \times \frac{5}{6}\right) - \left(\frac{2}{3} \times \frac{1}{6}\right)$

Do you think that multiplication is distributive over subtraction?

C. Sally had a piece of ribbon $3\frac{5}{8}$ yards long. She cut $1\frac{1}{4}$ yards from the piece to trim a dress. How many yards of ribbon were left?

D. The boys in Henry's club are to hike to Forest Lake, a distance of $7\frac{1}{5}$ miles. They plan to stop for lunch at a picnic area $3\frac{1}{4}$ miles from their starting point. How many miles will be left to hike?

E. Ira had a piece of board $10\frac{1}{2}$ feet long. He cut two pieces from the board each $4\frac{5}{8}$ feet long for book shelves. How many feet of board were left?

F. Lillian bought a pound of butter. She used $\frac{1}{4}$ of the pound of butter for a cookie recipe, and $\frac{1}{3}$ of the pound of butter to make a pudding What part of the pound of butter did she have left?

G. Mrs. Rivera had 4 yards of silk material. She used $\frac{3}{4}$ of a yard for a blouse and then used $\frac{1}{2}$ of the remaining material for a skirt.

How much material was used for the skirt?

H. Which number, $4\frac{2}{3}$ or $30\frac{1}{2}$, is farther from $17\frac{3}{5}$ on the number line?

III. Summary

A. How do we compute the difference of rational numbers expressed with the same denominator ?

B. How do we compute the difference of rational numbers expressed with different denominators?

C. Is subtraction of rational numbers commutative? Explain.

D. Is subtraction of rational numbers associative? Explain.

Lesson 68

Topic: Introduction to Ratio

Aim: To learn some basic concepts of ratio

Specific Objectives:

A ratio as a correspondence between the numbers of two sets of objects
How to express a ratio; symbols for ratio; terms of a ratio
A ratio as an ordered pair of numbers

Challenge: In what way are these statements alike?

a. I bought two pieces of candy for four pennies.
b. I made two model planes in four days
c. I baked two small pies for four guests.

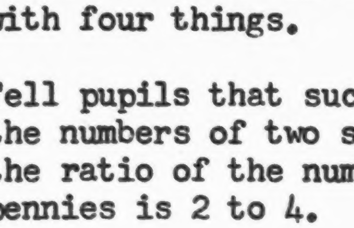
I. Procedure

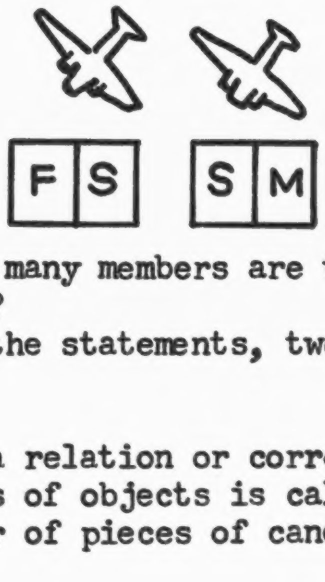
A. Meaning of ratio

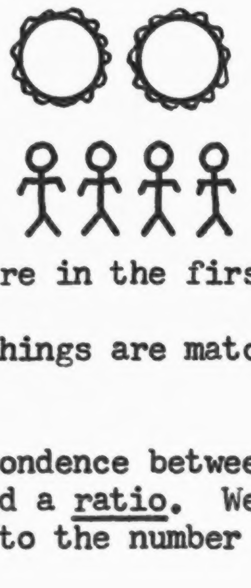
1. Elicit that two sets are described in each statement in the challenge. In the first statement, one of the sets is a set of pieces of candy. The other set is a set of pennies.

   What are the two sets in statement b? in statement c?

2. In statement a, two candies are matched with four pennies. What sets are matched in statement b? in statement c?







3. In each statement, how many members are there in the first set? in the second set?
   Elicit that in all of the statements, two things are matched with four things.

4. Tell pupils that such a relation or correspondence between the numbers of two sets of objects is called a ratio. We say the ratio of the number of pieces of candy to the number of pennies is 2 to 4.

How can we describe the correspondence between the number of model planes and the number of days in statement b? between the number of pies and the number of guests in statement c?

B. How to express a ratio

1. Present the symbols for ratio. For example, the way that we express the ratio 2 to 4 is 2:4 or $\frac{2}{4}$. Read this as "two to four."

   Elicit that two numerals are needed to express a ratio.

   In the ratio 2:4, the 2 and the 4 are called the terms of the ratio.

2. Have pupils read the ratios:

   a. 4:8 b. 9:12 c. 10:3 d. 1:2 e. $\frac{5}{6}$

   What are the terms of each ratio?

   Have pupils suggest some possible interpretations of these ratios. For example, the ratio 4:8 may describe the correspondence between four books to be used by eight pupils.

C. Order of the numbers in a ratio

1. How do the ratios 2:4 and 4:2 differ in meaning? Have pupils consider the following situations:

   a. Ann said, "I can read two books in four weeks."

      1) What sets are being compared? (a set of books, a set of weeks)

      2) What numerals would you use to express this ratio?

         (2:4 or $\frac{2}{4}$)

   b. If Ann said, "I take four weeks to read two books", the ratio would be 4:2 or $\frac{4}{2}$.

      1) What does the first number, 4, represent? (the number of things in the set which is now the first set, that is, the number of weeks)

      2) What does the second number, 2, represent? (the number of things in the set which is now the second set, that is, the number of books).

3) **Elicit that the order of the numbers in a ratio has meaning. Thus, we can think of a ratio as an ordered pair of numbers.**

2. Have pupils explain the difference in meaning between:

   a. 1:2 and 2:1

   b. 3:4 and 4:3

   c. 7:5 and 5:7

   d. $\frac{7}{8}$ and $\frac{8}{7}$

   e. $\frac{10}{3}$ and $\frac{3}{10}$

II. Practice

A. In what way are these situations alike?

   1. Helen walks 3 miles in an hour.

   2. There are three cookies for 1 boy.

   3. Three tomatoes are to be packed in 1 carton box

B. What are the two sets that are being compared in each statement in A?

C. In each of the following, state the two sets that are being compared. Write the symbol for the ratio that compares the two sets.

   1. The price is fifteen cents for two apples.

   2. John made four hits for seven times at bat.

   3. Mary won two out of three games.

   4. Bill finished four problems in the same time that Tom finished five problems.

   5. Paula ate lunch at school three out of the last four days.

D. Read the ratios: 1:4 2:9 13:5 $\frac{12}{2}$

E. What are the terms of each of the ratios in D?

III. Summary

A. What is meant by a ratio?

B. What symbols are used to express the ratio of two numbers?

C. What is meant by the statement: "A ratio is an ordered pair of numbers."

D. What new vocabulary have we learned today? (ratio, terms of a ratio, ordered pair)

Lesson 69

Topic: Ratio

Aim: To extend the meaning of ratio

Specific Objectives:

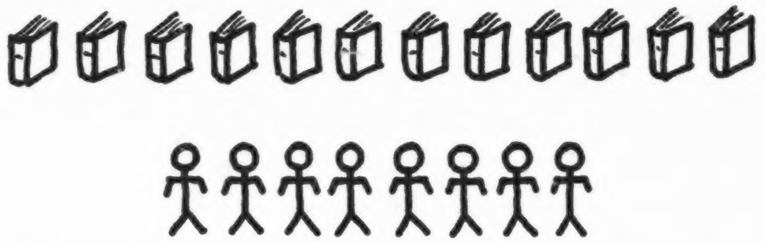
Names for a ratio
Expressing a ratio in "simplest form"

Challenge: There are 12 library books on mathematical topics which 8 members of the Mathematics Club plan to use for a research project. What is the ratio of the number of books to the number of pupils? Is there another name for this ratio? If so, how can we find it?

I. Procedure

A. Names for a ratio

1. Refer to the challenge problem. Have pupils diagram the problem as follows:



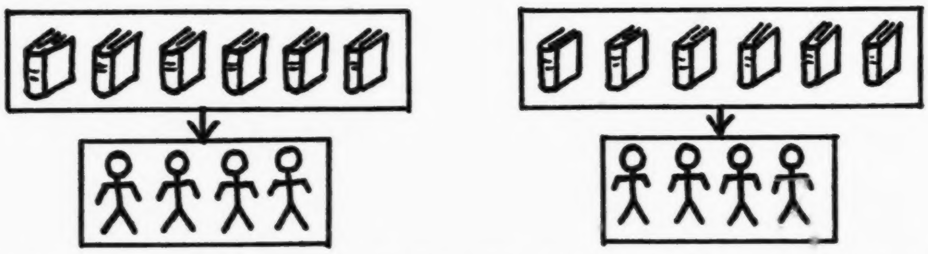
2. How can we express the correspondence between the number of the set of books and the number of the set of pupils as a ratio? (12:8)

3. Guide pupils to see that the members of the original sets may be rearranged in various ways.

   a. If we were to rearrange the set of books into two equivalent subsets, how many books would there be in each subset? (6)

   If we then rearrange the set of pupils into two equivalent subsets, how many pupils are there in each subset? (4)

Refer to the diagram:



1) What is the correspondence between the numbers of a subset of books and a subset of pupils? (6:4)

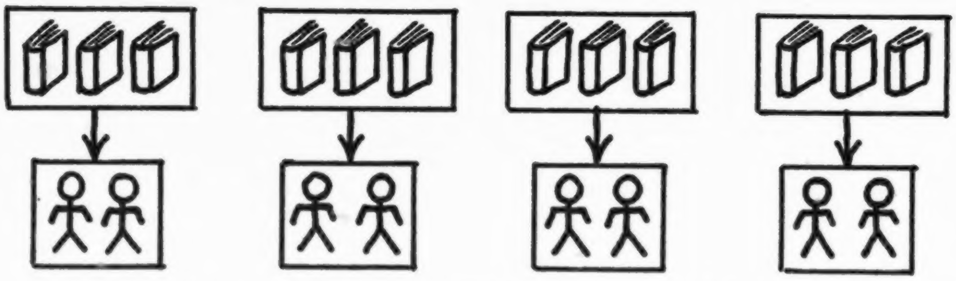
2) Elicit that this shows that for every set of 6 books in the original set of books, there corresponds a set of 4 pupils.

3) Thus, the correspondence between the original set of books and the original set of pupils may be expressed as 6 to 4 (6:4), as well as 12 to 8 (12:8).

b. If we were to rearrange the set of 12 books into four equivalent subsets, how many books would there be in each subset? (3)

If we then rearrange the set of pupils into 4 equivalent subsets, how many pupils are there in each subset? (2)

Refer to the diagram:



1) What is the correspondence between the numbers of a subset of books in this arrangement and a subset of pupils? (3:2)

2) Elicit that this shows that for every set of 3 books in the original set of books, there corresponds a set of 2 pupils.

3) Thus, the correspondence between the number of the original set of books and the number of the original set of pupils may be expressed as 3:2, as well as 6:4, as well as 12:8.

4. Elicit that a ratio such as 12:8 has several names. These names may be found by forming the same number of equivalent subsets in each of the original sets. Thus in the rearrangement considered in 3-a, each original set was arranged as two equivalent subsets; in the rearrangement considered in 3-b, each original set was arranged as four equivalent subsets.

5. Have pupils consider why we did not choose a rearrangement for each original set into three equivalent subsets.

   a. If a set of 12 books is arranged as three equivalent subsets, how many books are there in each set? (4)

   b. If a set of 8 pupils is arranged as three equivalent subsets, how many pupils are there in each set? Elicit that such an arrangement is not possible.

6. Have pupils use procedures such as those in 1 - 5 to find various names for ratios expressing the following situations:

   a. a set of 6 candy bars and a set of 4 boys

   b. a set of 4 bookshelves and a set of 48 books

   c. a set of 6 toys and a set of 3 children

   d. a set of 30 cookies and a set of 12 club members

B. Expressing a ratio in simplest form

1. Consider the challenge problem once more. How many elements are in the set of books? (12) How many elements are in the set of pupils? (8)

   a. What is a common factor of 12 and 8? (2) Into how many equivalent subsets were the original sets first rearranged? (2)

   b. What is another common factor of 12 and 8? (4) Into how many equivalent subsets were the original sets next rearranged? (4)

c. Elicit that a common factor can be used to tell us the number of equivalent subsets into which we may arrange each of the original sets.

2. What is the greatest common factor of 12 and 8? (4)
What is the name of the ratio when each of the original sets is rearranged into 4 equivalent subsets? (3:2)
What is the greatest common factor of 3 and 2? (1)

3. Tell pupils that a ratio is in its simplest form when its terms are relatively prime. Therefore, when we express the ratio of 12 to 8 as 3:2, we say that we have expressed it in lowest terms or in simplest form.

4. Have pupils give the simplest form for each of the ratios in A-6.

II. Practice

A. Give as many names as you can for the ratio that describes each of the following:

1. A rectangular flower bed is 20 feet long and 16 feet wide. What is the ratio of its length to its width?

2. Bob can ride to school on his bicycle in 6 minutes. It takes him 18 minutes to walk to school. What is the ratio of the time it takes him to walk to school to the time it takes him to ride?

3. 12 books for every 16 pupils

4. 15 packages mailed every 9 days

5. 20 gallons per 36 seconds

B. Circle the ratio that does not describe the correspondence between the numbers of the two sets of objects in each of the following:

1. 9 books for every 15 pupils: 3:5   8:12   9:15

2. 6 cans for 40¢: 20:3   6:40   3:20

3. 10 yards of material for 8 aprons: 5:4   10:8   8:6

C. Express each of the following ratios in simplest form:

1. 12:21
2. 10:6
3. 27:18
4. 8:10
5. 4:9
6. 36:20
7. 22:11
8. 15:25
9. $\frac{35}{10}$
10. $\frac{14}{42}$

III. Summary

A. If a ratio has more than one name, how would you go about finding another name for it?

B. What is meant by the simplest form of a ratio?

C. How do you find the simplest form of a ratio?

Lesson 70

Topic: Ratio

Aim: To learn how ratios and fractions are related

Specific Objectives:

To understand that although a ratio is a relation between the numbers of two sets, the way a ratio is expressed may not tell us how many elements are in each of the two sets

To understand why a ratio may be expressed by a fraction

Finding the simplest form for a ratio by expressing the ratio by a fraction and then finding the simplest form of the fraction

Challenge: In a certain school, the ratio of the number of girls to the number of boys is 3 to 2. How many girls and how many boys are there in the school?

I. Procedure

A. What a ratio tells us

1. Have pupils consider the following matchings of squares to circles:

Figure 1	Figure 2	Figure 3
□□□	□□□□□□	□□□□□□□□□
○○○○	○○○○○○○○	○○○○○○○○○○○○

2. Have them determine the simplest form for the ratio of the number of squares to the number of circles in each figure.

Figure 1	Figure 2	Figure 3
□□□	□□□ □□□	□□□ □□□ □□□
○○○○	○○○○ ○○○○	○○○○ ○○○○ ○○○○

   a. What is the ratio of the number of squares to the number of circles in Figure 1? (3 to 4)

   b. What is the ratio of the number of squares to the number of circles in Figure 2? (6 to 8, or 3 to 4)

c. What is the ratio of the number of squares to the number of circles in Figure 3? (9 to 12, or 3 to 4)

3. Guide pupils to see that although the number of elements in each set of squares (and in each set of circles) is not the same, the ratio of the number of squares to the number of circles is the same.

4. Elicit that a ratio of 3 to 4 used to describe the correspondence between the numbers of two sets means that for every three elements of the first set, there are four elements of the second set. The ratio does not tell us, however, how many elements are in either set.

5. Discuss the challenge question.

B. Why a ratio may be named by a fraction

1. Refer to the diagram in A-1. In each figure, the number of squares is how many times the number of circles?

   Elicit that in Figure 1, the number of squares is $\frac{3}{4}$ the number of circles.

   Elicit that in Figure 2, the number of squares is $\frac{6}{8}$ or $\frac{3}{4}$ the number of circles.

   Elicit that in Figure 3, the number of squares is $\frac{9}{12}$ or $\frac{3}{4}$ the number of circles.

2. Consider the following pairs of sets.
   What is the ratio of the number of the first set in each pair to the number of the second?
   The number of elements in the first set of each pair is how many times the number of elements in the second?

   a. 2 boys and 5 girls (the ratio is 2:5 and the number of boys is $\frac{2}{5}$ the number of girls)

   b. 5 U.S. stamps and 7 foreign stamps

   c. 20 votes for Bill and 11 votes for Jim

   d. 2 hours by plane and 18 hours by train

   e. 3 ounces of butter and 7 ounces of cheese

f. 30 points for Jim and 18 points for Harry

g. a $2 weekly allowance for Martha and a $3 weekly allowance for Janet

3. Guide pupils to see that if we know the terms of a ratio, we obtain their quotient (a performed division) to find how many times the number of one set is the number of the other set. Therefore, a ratio is often named by the fraction representing this quotient.

C. Finding the simplest form of a ratio

1. What is the simplest form of the ratio 6 to 10?

   a. What fraction expresses this ratio? ($\frac{6}{10}$)

   b. How did we determine the fraction? (when the first number, 6, of the ratio is divided by the second number, 10, the quotient is $\frac{6}{10}$)

   c. What is the simplest form of the fraction $\frac{6}{10}$? ($\frac{3}{5}$)

   d. What is the simplest form of the ratio 6 to 10? (3 to 5)

2. What is the simplest form of the ratio 7 to $1\frac{1}{2}$?

   a. What fraction expresses this ratio? ($\frac{7}{1\frac{1}{2}}$)

   b. How did we determine this fraction?

   c. What is the simplest form of the fraction?

$$\frac{7}{1\frac{1}{2}} = \frac{7}{\frac{3}{2}} = \frac{7 \times \frac{2}{3}}{\frac{3}{2} \times \frac{2}{3}} = \frac{7 \times \frac{2}{3}}{1} = \frac{14}{3}$$

   d. The simplest form of the ratio 7 to $1\frac{1}{2}$ is 14 to 3.

3. After several similar examples, elicit that we find the simplest form of a ratio by dividing the first number by the second and expressing the quotient as a fraction in simplest form.

II. Practice

A. The ratio of the number of boys in a class to the number of girls is 2 to 3.

1. Can you tell how many boys are in the class and how many girls? Explain.
2. If there are 10 boys in the class, how many girls are there?
3. If there are 18 girls in the class, how many boys are there?
4. For this ratio, 2 to 3, what are some other possible numbers of boys and of girls?

B. A survey showed that the ratio of the number of people who preferred Brand X soap powder to the number who preferred Brand Y is 7 to 10.

1. The number of people who preferred Brand X is _____ times the number who preferred Brand Y.
2. The number of people who preferred Brand Y is _____ times the number who preferred Brand X.

C. Express each of the following ratios by a fraction in simplest form:

1. 4 to 7
2. 18 to 27
3. 6 to 16
4. 42 to 24
5. 8 to 4 $(\frac{2}{1})$
6. 36:135
7. $\frac{3}{5} : \frac{4}{5}$
8. $\frac{1}{2}$ to 8
9. $\frac{3}{4}$ to $1\frac{1}{2}$
10. 6:$3\frac{1}{2}$

III. Summary

A. If you know the ratio of the numbers of the elements of two sets, can you tell how many elements are in either set? Explain.

B. What does a ratio tell us about the numbers of the two sets being considered? (It tells us how many times the number of one set is the number of the other set.)

C. How can we find the simplest form for a ratio?

Lessons 71 and 72

Topic: Ratio and Measures

Aim: To learn how to use ratios to compare measures

Specific Objectives:

To learn how to use ratios to compare like measures
To learn how to use ratios to compare unlike measures (rates)

Challenge: John is 60 inches tall. His father is 6 feet tall.
What is the ratio of John's height to his father's height?

I. Procedure

A. Using ratios to compare like measures

1. Refer to challenge.

Is $\frac{60}{6}$ the ratio of John's height to his father's height?

Although it is possible to match the number of inches in John's height to the number of feet in his father's height, such a correspondence is not particularly useful, since the impression may be conveyed that John's height is ten times that of his father's.

a. Have pupils express 60 inches as an equivalent number of feet, that is, as 5 feet.

What is the ratio of John's height to his father's? ($\frac{5}{6}$)

From this ratio we can see that John's height is $\frac{5}{6}$ of his father's.

b. Have pupils also express 6 feet as an equivalent number of inches, that is, as 72 inches.

What is the ratio of John's height to his father's?
($\frac{60}{72}$ or, in simplest form, $\frac{5}{6}$).

c. Elicit that in each case the ratio of John's height to his father's height is the same.

2. After several such examples, elicit that if a ratio is used to compare two like measures, such as two linear measures or two measures of weight, we use the same unit of measure for both. Thus, when a ratio omits mention of units of measure, it is assumed that both measures are expressed in the same unit.

3. Express the ratio for each of the following in simplest form:

a. 5 inches to 1 foot	d. 1 hour to 20 minutes
b. 3 feet to 3 yards	e. 8 ounces to 2 pounds
c. 2 feet to 6 inches	

4. On a map, we may find the scale 1" = 200 mi. (This means 1 inch represents 200 miles.) What is the ratio that corresponds to this scale?

   Have pupils see that if we follow the procedure of changing to the same unit (inches) the ratio is 1:12,672,000. (1 mile = 63,360 inches)

   However, because we are interested in converting inches on the map to miles on the ground, it is more convenient to express the scale as 1" = 200 mi.

B. Using ratios to compare unlike measures (rates)

1. Have pupils consider what the following situations have in common:

   32 feet per second
   5 gallons every 4 seconds
   30 miles in 1 hour
   5 inches every 19 minutes

   a. Elicit that in each case, the two measures are not the same kind. Thus, in 32 feet per second, one measure is linear and the other is a measure of time. There is no common unit of measure for distance and time.

   b. Guide pupils to see that it is still possible to compare the number of feet to the number of seconds, as, for example, 32 feet to 1 second.

2. Tell pupils that a ratio that compares unlike measures is called a rate. In expressing a rate, both units of measure must be stated, as in feet per second or miles per hour.

3. Have pupils give other examples of ratios that are rates, e.g., $2 per pound; 5 pints for every 3 seconds.

4. Tell pupils that we do not express a rate in fractional form. We usually use the words per, for or every, to state the relationship between the two measures.

II. Practice

A. Express each of the following as a ratio in simplest form:

1. 2 inches to 8 inches	6. 1 hour 20 minutes to 15 minutes
2. 1 inch to 1 foot	7. 6 lb. 3 oz. to 4 lb. 2 oz.
3. 5 yards to 4 feet	8. 3 quarts to 2 gallons
4. 2 lb. to 12 oz.	9. 5 minutes to 10 seconds
5. $3\frac{1}{2}$ oz. to 1 lb.	10. 2640 feet to 1 mile

B. Express each of the following pairs of measures as a rate.

1. 90 miles in 3 hours (30 miles per hour)
2. 75 barrels in 5 hours (15 barrels per hour)
3. $2 for 8 pints
4. 3261 feet in 3 seconds
5. 100 strokes in 20 seconds

C. A railroad transports 75,000 tons of freight in 15 days. What is the average daily rate of freight shipment?

D. If your heart beats 4320 times in one hour, what is the average rate of heart beat per minute?

III. Summary

A. Before we can express the ratio of two like measures such as two measures of length, what must be true of the units of measure? (They must be the same.)

B. Why does a scale on a map use different units of the same kind of measure?

C. In what way is a rate a special kind of ratio?

D. What new vocabulary have you learned today? (rate)

CHAPTER VII

The lessons in this chapter suggest a meaningful approach to the solution of equations and inequalities. Fundamental concepts used in all future mathematical learning-the concepts of the open sentence, variable, replacement set, and solution set- are developed and named.

This section is the first one that focuses on the nature of statements and open sentences. The phrase open sentence is a general term that embraces not only equations but inequalities as well. The study of inequalities is now considered to be as important as the study of equations. There is widespread use of inequalities in the physical sciences, in industry, and in the social sciences. There is no need to present equations and inequalities separately. Pupils should find it as natural to work with one as with the other.

The development in this chapter makes use of the analogies between English sentence structure and sentences about number ideas. A variable is used in an open sentence in a way similar to the use of a pronoun in ordinary language. We cannot determine whether the sentence: "She is wearing a red dress" is true or false until we replace "She" with the name of a person. In the same way, we cannot determine whether the sentence $n < 10$ is true or false until we replace n with a numeral. These sentences are examples of open sentences. However, it is important that emphasis be given to the understanding that an open sentence is neither true nor false. Otherwise, some pupils may infer that every sentence containing a variable is an open sentence. This, of course, is not the case. The sentence: $x + 5 = 5 + x$ is always true since it is an illustration of the commutative property of addition.

Not only does a variable hold the place for a numeral, but the replacement set for the variable must also be known. It makes little sense to choose a replacement for a variable which is not a member of the replacement set. Thus, in the sentence: "It is the largest of the Great Lakes," where the replacement set for "It" is the set of Great Lakes, "The Atlantic Ocean" cannot be permitted as a replacement because it is not a member of the replacement set. Similarly, in the open sentence $x < 5$ where the replacement for x is $\{2,3,4,5,6,7\}$, $3\frac{1}{2}$ is not a permissible replacement for x.

An important subset of the replacement set is the solution set. By replacing the variable in an open sentence with each element of the replacement set in turn, we find that the resulting statements, in effect, separate the replacement set into two disjoint subsets: the set whose elements form true statements and the set whose elements form false statements. The former set is called the solution set of the sentence. At this grade level, pupils use only this "replacement" method to determine the solution set of an open sentence.

Increased awareness of the importance of knowing the replacement set is gained as pupils see how different replacement sets may change the solution set of a given sentence. For example, the open sentence $3x = 5$ will have the empty set as its solution set if the replacement set is the set of whole numbers. If the replacement set is the set of rationals, the solution set is $\{ \frac{5}{3} \}$.

We can represent a solution set by drawing a graph of the set. At this stage, pupils develop an understanding of how a number line may be used as a geometric model on which to graph solution sets of open sentences, both equations and inequalities. In a higher grade, the graph of a solution set will be drawn on the coordinate plane.

CHAPTER VII

OPEN SENTENCES

Lessons 73-77

Lesson 73

Topic: Open Sentences

Aim: To learn some basic concepts of statements and open sentences

Specific Objectives:

Meaning of a mathematical statement
Meaning of an open sentence
Sentences and equations; sentences and inequalities

Challenge: Consider the following sentences about numbers

1. $10^3 = 1000$
2. $4+8 = 13$
3. $5^2 < 24$
4. $13+9 = 22$
5. $16-9 = \Delta$

Which are true and which are false?

I. Procedure

A. Mathematical statements

1. **Elicit that it is possible to decide whether each of sentences 1-4 in the challenge question is true or false. Thus, sentences 1 and 4 are true, and sentences 2 and 3 are false.**

2. **Tell pupils that a number sentence which can be judged true or false is called a statement.**

3. **Have pupils write several number sentences which are statements.**

B. Open sentences

1. **Consider sentence 5 of the challenge: $16-9 = \Delta$.**

   a. **Is this sentence true or false?**
   b. **If Δ were replaced by 6, could you then decide whether the sentence is true or false?**
   c. **What replacement for Δ would make $16-9 = \Delta$ a true**

statement? a false statement?

d. Follow same procedure with $\Box - 9 = 5$ and $8 \times ? = 24$.

2. Elicit that

   a. when a symbol such as Δ, $\Box$, or ? is used in a sentence, the sentence generally cannot be judged true or false; ($\Box + 5 = 5 + \Box$ can be judged true. It is an illustration of the commutative property of addition.)

   b. an appropriate replacement for the symbol makes the sentence a statement and it is then possible to judge it true or false.

3. Tell pupils that sentences like $16 - 9 = \Delta$, $\Box - 9 = 5$, $8 \times ? = 24$, which cannot be judged true or false, are called open sentences.

C. Sentences and equations; sentences and inequalities

1. Consider the following number sentences:

   $3+6 = 9$ $\quad$ $4-2 = 2$ $\quad$ $2 + \Box = 8$ $\quad$ $\Delta - 5 = 10$

   a. How are these sentences alike? (all involve the "=" symbol)

   b. Write four more sentences which involve the "=" symbol.

2. Tell pupils that any sentence using the symbol "=" is called an equation.

   An equation may be a statement (true or false) or an open sentence.

   Have pupils write several number sentences which are equations.

3. Consider these sentences:

   $8 > 6$ $\quad$ $2 + 7 \neq 10$ $\quad$ $4\frac{1}{2} + \Box > 15$ $\quad$ $\Delta - 2 < 3$

   a. What symbol means "is not equal to"?
   b. What symbol means "is greater than"?
   c. What symbol means "is less than"?

4. Tell pupils that a number sentence which uses one of the symbols $\neq$, $>$, or $<$ is called an inequality.

   An inequality may be a statement (true or false) or an open sentence.

   Have pupils write several number sentences which are inequalities.

II. Practice

A. Tell whether each of the following sentences is a statement or an open sentence. If it is a statement, tell whether it is true or false.

1. $4 \times 9 = 36$
2. $7 \times 13 \neq 20$
3. $3 \times ? > 12$
4. $4 \times 0 = \Box$
5. $\Box + \Delta = 7$
6. $2\frac{1}{2} + 2\frac{1}{2} < 6$
7. $7.9 + \Box = 8$
8. $8 \div ? = 4$

B. Select the sentences which are equations. Tell the reason for your choice.

1. 1 mile = 5280 feet
2. $3 \times 6 = 9 \times \Box$
3. \$1 = 75¢
4. $\frac{1}{2} = 60\%$
5. $203 = (2 \times 10^2) + (3 \times 1)$
6. $\frac{5}{0} \neq 5$
7. $\frac{54}{6} > 6 + 4$
8. $2.08 + .07 = 2.015$
9. $\frac{1}{2} + \frac{3}{4} = \Box$
10. $3 \times 2 = 2 \times 3$
11. $4 + (6+2) = (4+6) + 2$
12. $\frac{1}{2} + \frac{3}{4} = 1\frac{1}{4}$
13. $3 + \Box > 6$
14. $4 + 6 \neq 24$
15. $4 + 8 < \frac{1}{4} \times 60$
16. $.57 > .6$
17. $4 \times (2\frac{1}{2} + 2\frac{1}{3}) \neq 10 + 9\frac{1}{3}$
18. $\frac{0}{9} + \frac{0}{1} > 0$
19. $7 < \Delta$
20. $\frac{1}{3} > \frac{1}{4}$

C. Tell whether each of the equations in B is a statement or an open sentence. If it is a statement, tell whether it is true or false.

D. Tell whether each of the inequalities in B is a statement or an open sentence. If it is a statement, tell whether it is true or false.

E. Change the false statements in B to true statements.

F. Replace each ○ with =, >, or <, so that a true statement results.

1. $4 + 2 \bigcirc 2 \times 3$

2. $10 - 4 \bigcirc 2 + 8$

3. $0 \times 5 \bigcirc 1 + 0$

4. $7 \times 5 \bigcirc 5 \times 7$

5. $2\frac{1}{2} + 5 \bigcirc 9 - 2$

6. $99 + 1 \bigcirc 100$

7. $10 \times 100 \bigcirc 10{,}000$

8. $3.5 - 1.5 \bigcirc 2$

9. $146 + 28 \bigcirc 194$

10. $8 + 0 \bigcirc 8 \times 1$

G. Replace each $\square$ so that a true statement results.

1. $50 = 32 + \square$

2. $19 < \square + 10$

3. $\square = 5 \times 7$

4. $2 + \square > 10$

5. $30 + 40 + 1 = \square$

6. $\square - 60 = 485$

H. Using any or all of the numerals 2, 3 and 5

1. write four statements of inequality
2. judge each of them true or false

III. Summary

A. How do you decide whether a number sentence is a statement?

B. What do we mean by an open sentence?

C. When is a sentence considered an equation?

D. What symbols indicate that a sentence is an inequality?

E. What new vocabulary did you learn today?

(mathematical statement, open sentence, equation, inequality)

Lesson 74

Topic: Open Sentences

Aim: To learn to solve an open sentence by the replacement method

Specific Objectives:

Meaning of replacement set
Meaning of variable
Solving an open sentence by the replacement method

Challenge: Write a replacement for □ in the following sentence:

Earth is a □.

I. Procedure

A. Meaning of replacement set

1. Elicit several ways in which the challenge sentence can be completed, as, for example

   Earth is a planet.
   Earth is a sphere.

2. Elicit that each pupil who completes the sentence might be thinking of a set of things or ideas different from the set thought of by another pupil. Thus one pupil might think of {planet, star, meteor, satellite} as a set of possible replacements for □. Another might have in mind replacements which come from the set {sphere, cylinder, pyramid}.

3. Have pupils see that to avoid uncertainty, we should state the set of things or ideas we wish to consider as possible replacements for □. Such a set is called a replacement set. We must pick our replacements only from the replacement set.

4. Have pupils use the replacement set $\{1,2,3,4\}$ to replace the frame in $\square + 5 = 9$. They should replace □ by 1,2,3, and 4 in turn. Have them label each resulting statement true or false.

B. Meaning of variable

1. Consider the following open sentences:

   a. $3 \times \square = 15$ b. $n + 2 > 6$

2. Replace the frame in $3 \times \square$ using each member of the set $\{1, 1\frac{1}{2}, 5, 7\}$ in turn. Label each resulting statement either true or false.

3. Replace the n in $n + 2 > 6$ using each member of the set $\{3,4,5,6\}$ in turn. Label each resulting statement either true or false.

4. Elicit that the letters or frames in an open sentence may be replaced with any element of a specified replacement set. A letter or a frame used in this manner is called a variable.

5. What is the variable in the sentence in 1-a? in 1-b?

C. Solving open sentences

1. Consider the inequality $n + 3 > 5$ for which $\{1,2,3,4,5\}$ is the replacement set.

   a. Which replacements for the variable will make the resulting statements true?

      $1 + 3 > 5$ false
      $2 + 3 > 5$ false
      $3 + 3 > 5$ true
      $4 + 3 > 5$ true
      $5 + 3 > 5$ true

   b. The numbers 3,4,5 which make the inequality a true statement are called solutions of the inequality.

2. Consider the equation $x - 9 = 1$ for which $\{7,8,9,10,11\}$ is the replacement set.

   a. Which replacements for the variable result in true statements?

      $7 - 9 = 1$ false
      $8 - 9 = 1$ false
      $9 - 9 = 1$ false
      $10 - 9 = 1$ true
      $11 - 9 = 1$ false

   b. The only number from the replacement set which makes the equation a true statement is 1. The solution of the equation is 1.

3. The numbers from the replacement set which make an open sentence a true statement are called solutions of the open sentence. Finding these numbers is called solving the open sentence.

II. Practice

A. If the replacement set is the set of days of the week, which replacements for $\square$ would make the following sentence true?

$\square$ is a day whose name begins with a "T".

B. What is the variable in each of the following open sentences?

1. $10 + \square = 25$
2. $n - 4 = 13$
3. $\Delta \div 9 > 8$
4. $6.5 + 3.2 = x$

C. Write three equations using the variable n.

D. Write three inequalities using the variable x.

E. If the replacement set for the variable is the set of numbers, $\{0,1,2,3,4,\ldots 10\}$, find all the replacements that will make each sentence a true statement.

1. $x < 3$
2. $5 + n = 10$
3. $2 \times \square = 14$
4. $4 + x < 13$
5. $12 \div 3 = N$
6. $5\frac{1}{2} - 2\frac{1}{2} < \square$

F. Using the replacement set $\{1,2,3,4,5\}$, solve each of the following equations and inequalities.

1. $n + 6 = 10$
2. $n < 6$
3. $x - 3 > 1$
4. $10 - \square = 4$
5. $y - 1 = 3$

G. Solve each of the equations and inequalities in F using $\{0,2,4,6,8,10\}$ as the replacement set.

III. Summary

A. What is meant by a replacement set for an open sentence?
B. What is the meaning of a variable?
C. Which members of the replacement set are called solutions of an open sentence?
D. What new vocabulary did you learn today?
(replacement set, variable, solution)

Lesson 75

Topic: Open Sentence

Aim: To find the solution set of an open sentence

Specific Objectives:

Meaning of solution set
Finding the solution set of an open sentence
The number of elements a solution set may contain

Challenge: Is $4\frac{1}{2}$ a solution of the inequality $n > 3$?

I. Procedure

A. Meaning of solution set

1. Why is it not possible to tell whether $4\frac{1}{2}$ is a solution of the inequality $n > 3$? (no replacement set is given)

2. Suppose the replacement set for $n > 3$ is $\{2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}\}$. What are the solutions of the open sentence?

   a. Replace the variable by each element of the replacement set.

   $2 > 3$ false
   $2\frac{1}{2} > 3$ false
   $3 > 3$ false
   $3\frac{1}{2} > 3$ true
   $4 > 3$ true
   $4\frac{1}{2} > 3$ true

   b. Elicit that $3\frac{1}{2}$, 4, and $4\frac{1}{2}$ are the only solutions of the open sentence for the given replacement set.

3. Tell pupils that all of the members of a replacement set which make an open sentence true (that is to say, the solutions of the open sentence) make up the solution set of the open sentence. Thus, $\{3\frac{1}{2}, 4, 4\frac{1}{2}\}$ is the solution set of the open sentence $n > 3$.

4. Elicit that since every member of the solution set is a member of the replacement set, the solution set of an open sentence is a subset of the replacement set.

B. Finding the solution set of an open sentence

1. Consider the inequality $x + 1 < 5$ for which the replacement set is $\{0,1,2,3,4,5\}$. What is the solution set?

a. Replace the variable with each element of the replacement set.

$0 + 1 < 5$ true
$1 + 1 < 5$ true
$2 + 1 < 5$ true
$3 + 1 < 5$ true
$4 + 1 < 5$ false
$5 + 1 < 5$ false

b. Since the only members of the replacement set which make the open sentence true are 0,1,2, and 3, the solution set is $\{0,1,2,3\}$.

2. Consider the equation $3 + n = 10$ for which the replacement set is $\{5,6,7,8\}$. What is the solution set?

a. Replace the variable with each element of the replacement set.

$3 + 5 = 10$ false
$3 + 6 = 10$ false
$3 + 7 = 10$ true
$3 + 8 = 10$ false

b. The solution set is $\{7\}$.

3. Have pupils practice finding solution sets of equations and and inequalities for which the replacement sets are specified.

4. Suppose the replacement set for the open sentence in 1 is $\{1,3,5\}$. What is the solution set? ($\{1,3\}$)

Have pupils see that the solution set of an open sentence may not always be the same, depending upon the replacement set used.

C. The number of elements a solution set may contain

1. Consider the open sentence $x + 2 > 8$ with $\{1,2,\ldots,10\}$ as the replacement set.

a. What is the solution set? ($\{7,8,9,10\}$)

b. How many elements are there in this solution set?

2. Suppose the replacement set for the open sentence above is $\{1,2,3,4,\ldots\}$

a. What is the solution set? ($\{7,8,9,10,\ldots\}$)

b. Is there a greatest element of this solution set?

c. Elicit that this solution set is an infinite set.

3. Suppose the replacement set for the open sentence in 1 is $\{1,2,3,4,5\}$.

   a. What is the solution set?

   b. Elicit that since no members of the replacement set make the open sentence true, the solution set is the empty set.

4. Have pupils see that a solution set may be finite or infinite, or it may be the empty set.

II. Practice

A. Find the solution set of each open sentence if the replacement set is $\{1,2,3,\ldots,10\}$.

1. $3 \times n = 21$
2. $x < 3$
3. $x + 3 > 5$
4. $n + 1 = 9$
5. $a - 4 = 8$
6. $x - 5 < 7$

B. For the open sentence $1 + n > 3$, find the solution set that corresponds to each of the following replacement sets:

Replacement Set	Solution Set
1. $\{1,2,3,4,5\}$	
2. $\{1,2,3,4,5,\ldots\}$	
3. $\{1,2,3\}$	
4. $\{2,4,6\}$	
5. $\{0,1,2\}$	

III. Summary

A. What is meant by the solution set of an open sentence?

B. How is the solution set of an open sentence related to the replacement set?

C. How many members may we find in the solution set of an open sentence?

D. Does every open sentence have a solution set? Explain.

Lesson 76

Topic: Open Sentences

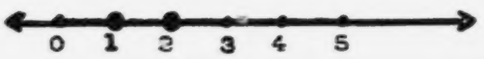
Aim: To learn to graph the solution set of an open sentence

Specific Objectives:

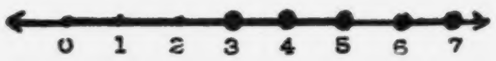
Review of concept of a number line
Graphing a number; a set of numbers
Graphing the solution set of an open sentence

Challenge: Consider the open sentence $n+3 > 5$ for which the replacement set is $\{0,1,2,3,\dots\}$. Which of the following may be thought of as a picture of the solution set?

(a)



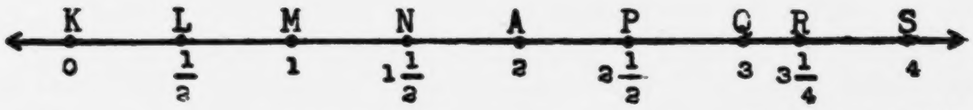
(b)



I. Procedure

A. Review concept of number line.

1. Have pupils draw a number line such as the following:



a. What point is associated with 0? with 2? with $3\frac{1}{2}$?

b. What number is associated with point M? with Q? with S?

c. Since $3 > 2\frac{1}{2}$, how do you expect point Q to be situated with respect to point P on the number line? (point Q is to the right of point P)

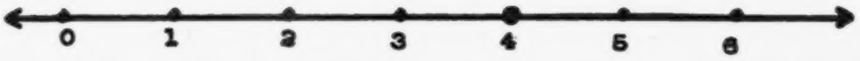
d. Consider the numbers associated with points to the left of the point paired with 4. How do these numbers compare in value with 4?

B. Graphing a number; a set of numbers

1. Recall the kinds of graphs the pupils have studied (line, bar).

2. What is a graph? (a picture of number relationships)

3. Tell pupils that on a number line, the point associated with a number is called the graph of the number. Thus, the graph of the number 4 is the point associated with 4. We show this graph by a darkened dot.



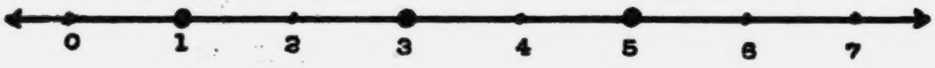
4. Have pupils draw a picture of a number line and show the graph of 2; of 7; of 10.

5. Consider the set {1,3,5}.

   a. How many elements does the set contain?

   b. How many points on the number line do you think should be darkened to show the graph of this set?

   c. Have pupils draw a number line and darken the dots denoting 1, 3 and 5.



   d. Tell pupils that the graph of the set {1,3,5} is the set of points (darkened dots) on the number line associated with these numbers.

6. Have pupils draw a picture of a number line and graph the following sets:

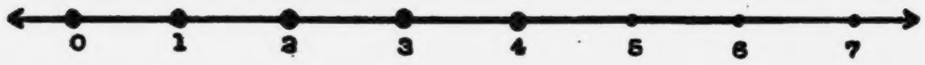
   a. {1,2,3}   b. {2,4,6,8}   c. {0,1,2,3,4,5}

C. Graphing the solution set of an open sentence

1. Consider the following open sentence for which the replacement set is the set of whole numbers. $4 \times n < 18$

   a. What is the solution set? ({0,1,2,3,4})

   b. Have pupils graph the solution set.

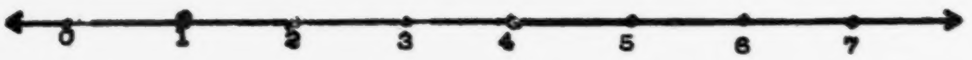


c. Tell pupils that the graph of the solution set of the open sentence is called the graph of the open sentence.

2. Consider the equation $n + 7 = 8$ for which the replacement set is the set of rational numbers (of arithmetic).

   a. What is the solution set? ($\{1\}$)

   b. Have pupils graph the solution set.



   c. Elicit that the darkened dot on the number line representing the point associated with the number 1 is the graph of the equation $n + 7 = 8$.

3. Refer to the challenge.

   a. What is the solution set? ($\{3,4,5,6,7,\ldots\}$)

   b. Elicit that the solution set is an infinite set. Figure (b) is a picture of the solution set.

   c. Have pupils see that since the last point named on the line is included in the graph, this shows that the whole numbers greater than 7 are also in the solution set.

4. Summarize the steps in graphing an equation (or inequality).

   a. Find the solution set of the equation (or inequality).

   b. Draw an appropriate number line.

   c. Locate the point(s) associated with the solution set.

   d. Indicate each point by a darkened dot.

II. Practice

A. Graph the following equations using the replacement set $\{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}, 5, 5\frac{1}{2}\}$.

1. $x = 5$
3. $3xy = 12$
2. $n+8 = 10\frac{1}{2}$
4. $4xn = 6$

B. Graph the following inequalities using the replacement set in A.

1. $x+7 > 10$

3. $4 < n+1$

2. $n-2 > 3$

4. $15 < 3 \times n$

III. Summary

A. What is the relation between numbers and points on a number line?

B. What steps do you follow when you graph an equation or an inequality?

C. What new vocabulary did you learn today?

(graph of an equation, graph of an inequality)

CHAPTER VIII

Some decimal numerals are simply names for the already familiar set of rational numbers. Thus, the arithmetic of rational numbers is applicable when performing computation involving decimal numerals. The lesson plans in this chapter suggest procedures for helping pupils develop this understanding and for extending their concepts and skills in operations with rational numbers named in decimal form.

This section provides for reinforcing the pupil's understanding of how our notation system is extended to the right of a decimal point. The systematic nature of this extension is emphasized. Pupils are thus guided to see that decimal fractions are a different and easy way of expressing fractions that have 10 or a power of 10 as denominators. For example, a decimal such as .3 and the common fraction $\frac{3}{10}$ are two names for the same number. Just as other common fractions name $\frac{3}{10}$ (e.g., $\frac{6}{20}$, $\frac{12}{40}$, etc.) so other decimals, such as .30, .300, .3000, can be used to name this number.

One of the objectives of this chapter is to suggest procedures for helping the pupil review and extend the process called "rounding numbers." Thus, the pupil will be able to use this understanding intelligently as he works with measures and with estimating answers.

The important reason for studying decimal numerals is that frequently an operation is easier to perform in decimal form than in fractional form. Addition and subtraction of fractional numbers is much simpler in decimal form if these numbers have terminating decimal representations. Pupils should be guided to see the advantage of the decimal form over the fractional form as they contrast, for example, the computation involved in $4\frac{73}{100} + 15\frac{85}{100}$ with that in 4.73 + 15.85.

There is strong emphasis in these materials on rationalizing the algorithms that are used in computation with decimal numerals. It is expected that a better appreciation of our numeration system will result.

Since the operations of addition and subtraction with decimals were developed in earlier grades, the review, reinforcement, and extension of these concepts and skills will generally not take too much time.

The operation of multiplying rational numbers represented by decimal numerals is approached from a consideration of the pupil's past experiences with rational numbers represented by common fractions. The procedure of counting the number of places following the decimal points in the two factors is a mechanical process, but it is very convenient. The pupil should, however, understand why this technique works.

Division of rational numbers represented by decimal numerals is considered from the standpoint of transforming the numerals so that the quotient remains the same but the divisor is a natural number. Procedures are suggested for developing pupil understanding of why the division algorithm may be extended to rational numbers which are represented as decimal numerals. Thus, in 1.00 ÷ 4 = .25, it is important that pupils understand that 1.00 may be thought of as representing 100 hundredths, just as 100 may be viewed as representing 100 ones, or 10 tens, or 1 hundred. Then 100 hundredths divided by 4 is 25 hundredths or .25.

Throughout the material in this chapter, opportunity is provided for the pupil to use decimal numerals in applied problems.

Chapter VIII

RATIONAL NUMBERS: DECIMAL FORM

Lessons 77-85

Lessons 77 and 78

Topic: Decimal Notation

Aim: To extend understanding of decimal notation and of decimal fractions

Specific Objectives:

Extending decimal notation
Equivalent decimals
Comparing decimals

Challenge: Which names the greater number, .029 or .3?

I. Procedure

A. Extending decimal notation

1. Review table of place values

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths

a. How many times the value of ones place is tens place? of tens place is hundreds place, etc.?

As we move to the left, how many times the value of the place to its right is a given place value?

b. What part of the value of tens place is ones place? of ones place is tenths place? of tenths place is hundredths place?

As we move to the right, what part of the value of the place to its left is any given place value?

c. How many places to the left of the ones place is the thousands place?

How many places to the right of ones place is the thousandths place?

2. Extend place value table to the right.

a. What part of the value of thousandths place is the place to its right? ($\frac{1}{10}$)

b. What is the value of this place?

$$\frac{1}{10} \times \frac{1}{1000} = \frac{1}{10,000} \text{ (one ten-thousandth)}$$

Note: Enter ten-thousandths place in table.

c. How many places to the right of ones place is the ten-thousandths place?

How many places to the right of the decimal point does a ten-thousandths decimal have?

d. What is the value of the fifth place to the right of the decimal point? the sixth place?

Note: Enter hundred-thousandths and millionths places in table.

How many places to the right of the decimal point does a hundred-thousandths decimal have? a millionths decimal?

3. Have pupils recall that the value of a digit in a numeral is the value of the number named by the digit times the value of the place.

a. What is the value of the digit 2 in .002358?
b. What is the value of the digit 3 in .002358?
c. What is the value of the digit 5 in .002358?
c. What is the value of the digit 8 in .002358?

4. Have pupils practice reading and writing decimal numerals.

a. What is the word name for each of the following?

1) .0005
2) .0038
3) .0162
4) .2347
5) .00035
6) .01756
7) .000038
8) .195060

b. Give decimals for word names.

1) eight ten-thousandths
2) one thousand fifty-seven ten-thousandths
3) one and fourteen ten-thousandths
4) thirteen thousand, one hundred fifteen hundred thousandths
5) seven hundred eighteen millionths

5. Have pupils write fractional names for each of the decimals in 4-a and in 4-b.

a. $.0005 = \frac{5}{10000}$ or $\frac{1}{2000}$ in simplest form

b. $.0038 = \frac{38}{10000}$ or $\frac{19}{5000}$ in simplest form, and so on

B. Equivalent Decimals

1. Recall that a number may be named in many ways.

a. $.2 = \frac{2}{10} = \frac{2 \times 10}{10 \times 10} = \frac{20}{100} = .20$ (20 hundredths)

b. $.2 = \frac{2}{10} = \frac{2 \times 100}{10 \times 100} = \frac{200}{1000} = .200$ (200 thousandths)

c. $.2 = \frac{2}{10} = \frac{2 \times 1000}{10 \times 1000} = \frac{2000}{10,000} = .2000$ (2000 ten-thousandths)

d. $.2 = \frac{2}{10} = \frac{2 \times 10,000}{10 \times 10,000} = \frac{20000}{100,000} = .20000$ (20000 hundred-thousandths)

e. $.2 = \frac{2}{10} = \frac{2 \times 100,000}{10 \times 100,000} = \frac{200,000}{1,000,000} = .200000$ (200,000 millionths)

Thus, .2 = .20 = .200 = .2000 = .20000 = .200000

f. .2, .20, .200, .2000, .20000, .200000 are said to be equivalent decimals.

They are different decimal names for the same number.

2. What property of numbers have we used in renaming .2? (multiplication property of one)

3. Have pupils change .5 to hundredths; to thousandths; to ten-thousandths; to millionths.

C. Comparing decimals

1. Refer to challenge.

   a. Since .029 is expressed as thousandths, let us rename .3 as thousandths. $.3 = .300$

   b. Compare .029 with .300 rather than .029 with .3.

   c. Elicit that since we know $.029 < .300$, then we know that $.029 < .3$.

2. Have pupils see that to compare two decimals of different denominations, we may first express them in the same denomination, and then make the comparison.

   Have them recall that a similar procedure was used in comparing numbers expressed in fractional form.

II. Practice

A. What is the value of the digit 2 in .2146? of the digit 1? of the digit 4? of the digit 6?

B. Replace the frames.

$$.2146 = \frac{2}{10} + \frac{1}{100} + \frac{\square}{1000} + \frac{6}{\Delta}$$

C. Give word names for decimals.

1. .146
2. .1968
3. .03922
4. .000145

D. Give decimals for word names.

1. seventeen thousandths
2. six hundred twenty-two ten-thousandths
3. eighty-seven thousand, six hundred two hundred-thousandths
4. nine hundred twenty-five millionths

E. Rename as equivalent decimals in thousandths.

1. .80
2. .1
3. $\frac{8}{10}$
4. five hundredths

F. Rename as equivalent decimals in hundredths.

1. .4
2. .610
3. .9200
4. .30000

G. Which one of the following is not another name for .6?

1. .6000
2. .60
3. .0600

H. Express each of the following decimals as common fractions in simplest form.

1. .013
2. .145
3. .0036
4. .1250
5. .000175

I. Consider the following pairs of decimals. In each case, which represents the greater number?

1. .5, .46
2. .47, .168
3. .2222, .33
4. .09641, .1001
5. .0302, .203

J. Find the solution set of each equation or inequality, using the replacement set given at the right.

1. $x < .35$ $\{.1, .4, .25, .345\}$
2. $x > .7$ $\{.5, .55, .70, .705\}$
3. $x = .100$ $\{.01, .001, .1\}$

III. Summary

A. What is the value of the fourth place to the right of the decimal point? the fifth place? the sixth place?

B. What is the value of the digit 8 in .000814? of the digit 1? of the digit 4?

C. What is meant when we say that two decimals are equivalent?

D. What property of numbers do we use in changing a decimal to an equivalent decimal? Explain.

E. How may we compare two decimals which are expressed in different denominations?

F. What new vocabulary did you learn today?

(ten-thousandths, hundred-thousandths, millionths, equivalent decimals)

Lesson 79

Topic: The Decimal Form of Rational Numbers

Aim: To express a rational number in decimal form

Specific Objectives:

Review of meaning of rational numbers
Expressing (some) rational numbers in decimal form

Challenge: Which of these are names for rational numbers?

$$5 \qquad \frac{3}{4} \qquad 2\frac{1}{2} \qquad 1.5$$

I. Procedure

A. Review rational numbers expressed by fractions. (See Chapter VI)

1. Elicit that each of the numbers named in the challenge is a rational number because each can be expressed as a quotient of a whole number by a counting number. Thus,

$$5 = \frac{5}{1} \qquad 2\frac{1}{2} = \frac{5}{2} \qquad 1.5 = 1\frac{5}{10} \text{ or } \frac{15}{10}$$

2. Is $\frac{7}{0}$ a name for a rational number? Explain.

3. Write several fractions that symbolize zero.

4. Show that many fractions name the rational number 3.

B. Expressing rational numbers in decimal form

1. Rename the following in fractional form.

a. .79

b. .293

c. .0418

d. .23456

e. .000375

Have pupils see that remembering the value of each place in a decimal enables us to change a decimal to an equivalent fractional numeral.

2. Why is each of the numbers represented in 1 above a rational number?

3. Change to decimal form:

   a. $\frac{6}{10}$ b. $\frac{18}{100}$ c. $\frac{625}{1000}$ d. $\frac{7}{100}$

   e. $\frac{7}{1000}$ f. $\frac{8629}{10000}$ g. $5\frac{15}{100}$ h. $23\frac{9}{10}$

   Elicit that when a rational number is expressed by a fraction with a denominator of 10, 100, 1000, 10000, we can immediately rename the number in decimal form.

4. Have pupils recall that 10, 100, 1000, 10,000 are powers of 10.

5. Suppose a rational number is expressed by a fraction which has a denominator that is not a power of 10. How can we rename it in decimal form?

   a. Consider $\frac{1}{2}$. How can we change $\frac{1}{2}$ to decimal form?

      1) Is there an equivalent fractional name for $\frac{1}{2}$ with a power of ten as a denominator?

      2) Try the first power of 10. Is 10 divisible by 2?

      3) $\frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10} = .5$

      4) Other decimal names for $\frac{1}{2}$ are: .50, .500, .5000.

   b. Change $\frac{1}{4}$ to decimal form.

      1) Try the first power of 10. Is 10 divisible by 4?

      2) Try the second power of 10. Is 100 divisible by 4?

      3) $\frac{1}{4} = \frac{1 \times 25}{4 \times 25} = \frac{25}{100} = .25$

      4) What are other decimal names for $\frac{1}{4}$?

   c. Name $\frac{3}{8}$ in decimal form.

      1) Is 8 a factor of 10? of 100? of 1000?

      2) $\frac{3}{8} = \frac{3 \times 125}{8 \times 125} = \frac{375}{1000} = .375$

      3) What are other decimal names for $\frac{3}{8}$?

6. Elicit that to find a decimal name for a rational number expressed by a fraction which has a denominator that is not a power of 10, we must first find an equivalent fractional name with a power of 10 as a denominator.

II. Practice

A. Change fractional numerals to equivalent decimals. Change decimals to equivalent fractional numerals.

1. $\frac{4}{10}$
2. .26
3. .170
4. $\frac{35}{1000}$
5. .0023
6. $\frac{1765}{10,000}$
7. $\frac{50000}{100,000}$
8. $\frac{950}{1,000,000}$

B. Why can we easily write the fractional numerals in A as equivalent decimals? (the denominators are powers of 10)

C. For each of the following tell which power of 10 must be used to change to decimal form.

1. $\frac{3}{5}$
2. $\frac{7}{20}$
3. $\frac{3}{4}$
4. $\frac{19}{50}$
5. $\frac{5}{8}$
6. $\frac{15}{40}$
7. $\frac{25}{80}$
8. $\frac{3}{16}$
9. $1\frac{1}{5}$
10. $3\frac{1}{8}$
11. $\frac{1}{200}$
12. $\frac{3}{400}$

D. Change each of the numerals in C to decimal form.

III. Summary

A. What is a rational number?

B. Give an illustration of a rational number in fractional form; in decimal form.

C. How do we rename in decimal form a rational number expressed by a fraction whose denominator is not a power of 10?

Note to Teacher: The procedure developed in this lesson for renaming in decimal form (as a terminating decimal) a rational number expressed by a fraction whose denominator is not a power of ten does not apply to all fractions, as, for example, $\frac{1}{3}$, $\frac{1}{9}$, and so on.

Lesson 80

Topic: The Decimal Form of Rational Numbers

Aim: To learn to round fractional numbers named in decimal form

Specific Objectives:

Review (if necessary) of rounding whole numbers
Rounding to the nearest whole number; to the nearest tenth
Rounding fractional numbers named in decimal form as indicated

Challenge: A newspaper reported that the attendance at a school basketball game was 400. Why did the newspaper use a "rounded" number in reporting the attendance?

I. Procedure

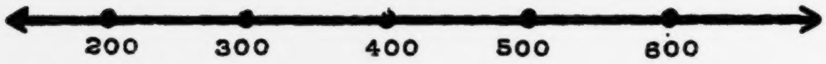
A. Review rounding whole numbers

Note: If pupils have understanding and skill in rounding whole numbers, the material in this section may be omitted.

1. Elicit that the exact number of people who were at the basketball game might not have been easy to determine and, in fact, is not needed. To give the reader of the newspaper an idea of the size of the crowd, a rounded number is sufficient.

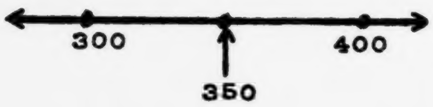
2. If there are 375 seventh grade pupils in a school, what is the number of pupils to the nearest hundred?

   a. Study the number line below. Points on it are matched with hundreds which are near 375.

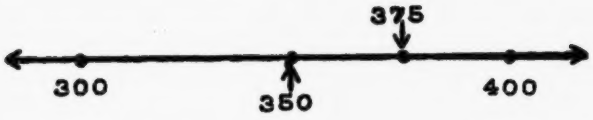


      1) Between which two points will the point to be paired with 375 lie?

      2) Mark the point halfway between the points paired with 300 and 400. Which number is matched with it?



    3) Will the point to be paired with 375 lie to the right or to the left of the halfway point?



    4) Is 375 closer to 300 or to 400?

  b. What is the number of seventh grade pupils in the aforementioned example rounded to the nearest hundred?

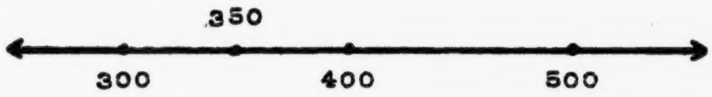
  c. Have pupils see that we can round 375 without using a number line.

    1) Subtract to find whether 375 is nearer to 300 or to 400.

       375 - 300 = 75   400 - 375 = 25

    2) Since the difference between 300 and 375 is greater than the difference between 375 and 400, we see that 375 is nearer to 400. Therefore, if 375 is rounded to the nearest hundred, it becomes 400.

3. Round 350 feet to the nearest hundred feet.



  a. Between which two successive hundreds does 350 fall?

  b. Is 350 closer to 300 or to 400?
     Have pupils recall that in cases like this we agree to round to the greater number.

  c. 350 rounded to the nearest hundred is 400.

4. Round the numbers named below to the nearest hundred.

  a. 238   d. 850

  b. 469   e. 749

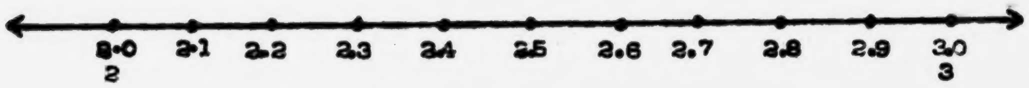
  c. 106

5. If necessary, use similar procedures to review rounding to the nearest ten; to the nearest thousand; to the nearest ten-thousand; and so on.

B. Rounding decimal fractions

1. Round 2.7 to the nearest whole number.

    a. Study the following number line.



    b. Between which two whole numbers is 2.7 located?

    c. Locate 2.7 on the number line.
    Is 2.7 closer to 2 or to 3?

    d. Elicit that to the nearest whole number we would round 2.7 to 3.

    e. Have pupils see that instead of using a number line, we can subtract to find whether 2.7 is closer to 2 or to 3.

        1) Rename 2 as 2.0, and rename 3 as 3.0.

        2) 2.7 - 2.0 = .7 3.0 - 2.7 = .3

        Which difference is greater?

        3) Since the difference between 2.7 and 2.0 is greater than the difference between 2.7 and 3.0, we say that 2.7 rounded to the nearest whole number is 3.

2. Round 3.5 to the nearest whole number.

    a. Between which two whole numbers is 3.5?

    b. Is 3.5 closer to 3 or to 4?

    c. Recall that if a number is halfway between two numbers, we shall agree to round to the greater number.

    d. To which whole number would you round 3.5?

3. Round to the nearest whole number.

    a. 4.6

    d. 46.9

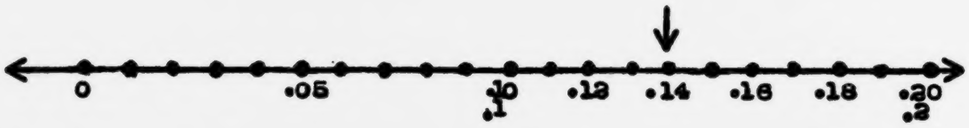
    b. 28.3

    e. 1.2

    c. 9.5

4. Round .14 to the nearest tenth.

   a. Study the following number line.



      1) Is .14 between 0 and .1, or between .1 and .2?

      2) Is .14 nearer to .1 or to .2?

      3) Round .14 to the nearest tenth.

   b. Have pupils subtract to find whether .14 is nearer to .1 or to .2.

      1) Rename .1 as .10, and rename .2 as .20.

      2) .14 - .10 = .04 .20 - .14 = .06

         Which difference is greater?

      3) Round .14 to the nearest tenth.

C. Rounding to any place

1. Have pupils see that rounding to any place can be done in the same way as rounding to tenths.

   a. Round 4.167 to the nearest whole number.

      1) Between which two whole numbers is 4.167 located? (4 and 5)

      2) Rename 4 as 4.000, and rename 5 as 5.000.

      3)

$$\begin{array}{r} 4.167 \\ -4.000 \\ \hline .167 \end{array} \qquad\qquad \begin{array}{r} 5.000 \\ -4.167 \\ \hline .833 \end{array}$$

         Which difference is greater?

      4) Round 4.167 to the nearest whole number.

b. Round .678 to hundredths.

1) Between which two hundredths numbers is .678 located? (.67 and .68)

2) Rename .67 as .670, and rename .68 as .680.

3) $$\begin{array}{r} .678 \\ -\underline{.670} \\ .008 \end{array} \qquad \qquad \begin{array}{r} .680 \\ -\underline{.678} \\ .002 \end{array}$$

Which difference is greater?

4) What is .678 rounded to hundredths?

2. Have pupils practice rounding as indicated.

II. Practice

A. Round each number named below to the nearest whole number.

1. 6.9
2. 20.2
3. 619.5
4. 1023.3
5. 482.1

B. Round each number named below to the nearest tenth.

1. 12.62
2. 4.15
3. 15.99
4. 200.78
5. 43.50

C. Round each number named below to the nearest hundredth.

1. .421
2. 3.885
3. 35.970
4. 1.011
5. 119.689

D. Round each number named in C to the nearest whole number; to the nearest tenth.

E. Round each number named below to the nearest thousandth.

1. .1975
2. 16.1515
3. 28.1489
4. 22.0099
5. 216.1234

F. Round each number named in C to the nearest hundredth; to the nearest tenth.

G. Write each of the following as a decimal. Then round to the nearest tenth.

1. $\frac{19}{100}$
2. $\frac{3}{4}$
3. $\frac{1}{5}$
4. $\frac{37}{50}$
5. $\frac{9}{25}$

H. Write each of the following as a decimal. Then round to the nearest hundredth.

1. $\frac{1}{8}$
2. $\frac{5}{8}$
3. $\frac{1}{16}$
4. $\frac{3}{16}$

III. Summary

A. Why do we round numbers?

B. How do we round a whole number to the nearest ten? to the nearest hundred?

C. How do we round a decimal number to the nearest whole number? to the nearest tenth? to any place?

D. What new vocabulary did you learn today?

(nearest tenth; nearest hundredth; nearest thousandth)

Lesson 81

Topic: The Decimal Form of Rational Numbers

Aim: To develop increased understanding and skill in addition and subtraction with decimals

Specific Objectives:

Addition with decimals
Subtraction with decimals

Challenge: The Weather Bureau reported the rainfall for two successive days as .87 and .64 inches. What was the total rainfall for the two days?

I. Procedure

A. Addition with decimals

1. Elicit that the challenge problem requires the operation of addition.

   a. $.87 + .64 = \frac{87}{100} + \frac{64}{100} = \frac{87+64}{100} = \frac{151}{100} = 1.51$

   b. Recall that addition with decimals may be shown in vertical form.

$$\begin{array}{r} .87 \\ +\underline{.64} \\ 1.51 \end{array}$$

2. Compute the sum of .283 and .469.

   a. $.283 + .469 = \frac{283}{1000} + \frac{469}{1000} = \frac{283+469}{1000} = \frac{752}{1000}$

   b. $\begin{array}{r} .283 \\ +\underline{.469} \\ .752 \end{array}$

3. Have pupils use the same pattern to compute the sum of .7068 and .1433.

   a. Replace the frames.

$$.7068 + .1433 = \frac{7068}{10{,}000} + \frac{\square}{10{,}000} = \frac{\triangle+1433}{10{,}000} = \frac{\frown}{10{,}000} = .8501$$

   b. Add .7068 and .1433 in vertical form.

4. Elicit that when we compute using the vertical form, we align the decimal points. This enables us to add tenths to tenths, hundredths to hundredths, and so on. In this way, addition of decimals is no more difficult than addition of whole numbers.

5. How do we compute the sum of two rational numbers whose decimal names have a different number of decimal places?

   a. Compute the sum of .6 and .24.

      Since .6 can be renamed as .60, we may compute the sum of .6 and .24 as follows:

      1) $.6 + .24 = .60 + .24 = .84$

      2) $$\begin{array}{r} .60 \\ +.24 \\ \hline .84 \end{array} \quad \text{or} \quad \begin{array}{r} .6 \\ .24 \\ \hline .84 \end{array}$$

      3) What would happen if the decimal points were not aligned?

   b. Elicit that renaming a decimal as an equivalent decimal enables us to compute the sum of two rational numbers whose decimal names have a different number of decimal places.

   c. Compute the sums.

      1) $.6 + .12$

      2) $.09 + .3$

      3) $1.2 + 4.58$

      4) $3 + .7246$

6. Have pupils see that we can also use the distributive property of multiplication over addition to show regrouping.

   a. $$\begin{aligned} .16 + .25 &= \frac{16}{100} + \frac{25}{100} \\ &= \left(16 \times \frac{1}{100}\right) + \left(25 \times \frac{1}{100}\right) \\ &= (16 + 25) \times \frac{1}{100} \\ &= 41 \times \frac{1}{100} \\ &= \frac{41}{100} \\ &= .41 \end{aligned}$$

   b. Elicit that the vertical form is the more convenient form to use.

B. Subtraction with decimals

1. Pose problem. A weather station measured the rainfall on a certain date this year and found it to be .85 inches. The rainfall on the same date last year was .23 inches.

   How much more rain fell on that date this year than last year?

   a. Elicit that the problem requires the operation of subtraction.

      1) $.85 - .23 = \frac{85}{100} - \frac{23}{100} = \frac{85-23}{100} = \frac{62}{100} = .62$

      2) Recall that subtraction with decimals may be shown in vertical form and performed in a manner similar to subtraction of whole numbers.

$$\begin{array}{r} .85 \\ -\underline{.23} \\ .62 \end{array}$$

      3) Why is it important to align the decimal points when we use the vertical form?

2. Compute the difference of .56 and .115.

   a. Rename .56 as .560 and compute the difference of .56 and .115 as follows:

$$\begin{array}{r} .560 \\ -\underline{.015} \\ .545 \end{array} \qquad \begin{array}{r} 5\,10 \\ .5\cancel{6}\cancel{0} \\ -\underline{.015} \\ .545 \end{array}$$

   b. Elicit that when writing the work in vertical form, we "exchange" (if necessary) in the same way as we do for subtraction of whole numbers.

3. Subtract:

   a. .86 - .32

   b. 4.09 - 1.50

   c. .87 - .108

   d. 3.19 - .1465

II. Practice

A. Write in vertical form. Then find the sums.

1. .960 + .096
2. 4.23 + 6.832 + 4.4
3. 17.9 + 32.8 + 46.5
4. 39.2855 + 93.703
5. 2.13141 + 21.3141

B. Subtract:

1. 3.65 - .24
2. .70 - .38
3. 15.82 - 12.0471
4. $5 - $1.97
5. 27.4813 - 13.7509

C. John kept a record of his gasoline purchases for his new car during one month: 17.8 gal.; 12.3 gal.; 8 gal.; 15.5 gal.; 6.4 gal.; 11.9 gal. How many gallons of gasoline were purchased during that month?

D. A transport company had the following amounts of freight to deliver: 18.5 tons of steel, 5.85 tons of groceries, 9.20 tons of lumber, and 48.6 tons of coal. How many tons of freight were to be delivered?

E. A machinist was making a metal part which was to be 1.635 inches thick. The piece of metal he was working with measured 1.925 inches. How many inches would he have to cut off?

F. Charles' batting average dropped from 0.425 at the end of the fifth game to 0.231 at the end of the season. What is the difference between these two batting averages?

III. Summary

A. How is addition with decimals similar to addition with whole numbers?

B. How is subtraction with decimals similar to subtraction with whole numbers?

C. How do we compute the sum (or difference) of two rational numbers whose decimal names have a different number of decimal places?

Lesson 82

Topic: Multiplication of Rational Numbers: Decimal Form

Aim: To extend understanding of and skill in multiplying rational numbers expressed in decimal form

Specific Objectives:

Reinforcement of finding a product when one of the factors is named by a decimal
Finding a product when both factors are named by decimals

Challenge: A piece of metal is .4 of an inch thick.
How high is a pile of 8 such pieces of metal?

I. Procedure

A. Reinforcement of finding a product when one of the factors is named by a decimal

1. Elicit that the challenge problem may be solved by the operation of multiplication.

a. Estimate: Since $.4 < .5$ or $\frac{1}{2}$, then $8 \times .4 < 8 \times \frac{1}{2}$, and $8 \times .4 < 4$.

b. $$\begin{aligned} 8 \times .4 &= 8 \times \frac{4}{10} \\ &= 8 \times (4 \times \frac{1}{10}) \\ &= (8 \times 4) \times \frac{1}{10} \quad \text{Why?} \\ &= 32 \times \frac{1}{10} \\ &= \frac{32}{10} \text{ or } 3.2 \quad \text{(Compare with estimate.)} \end{aligned}$$

c. Recall that the work may be shown in vertical form.

$$\begin{array}{r} .4 \\ \times\ \underline{8} \\ 3.2 \end{array}$$

d. Elicit that the product of 8 and .4 was computed in a way similar to finding the product of 8 and 4. However, since there is one decimal place in the numeral for the factor .4, there must be one decimal place in the numeral for the product.

2. Have pupils compute these products as in 1-b. Have them also show the work in vertical form.

a. $5 \times .13$ b. $.007 \times 6$ c. $8 \times .0015$

3. Elicit that the number of decimal places in the product numeral is the same as the number of decimal places in the factor named by a decimal.

   Note: Estimating the product before multiplying is helpful in locating the decimal point in the product numeral.

B. Finding a product when both factors are named by decimals

1. Consider $.4 \times .3 = \square$

$$\begin{aligned} .4 \times .3 &= \frac{4}{10} \times \frac{3}{10} \\ &= \frac{4 \times 3}{10 \times 10} \\ &= \frac{12}{100} \\ &= .12 \end{aligned}$$

Vertical Form

$$\begin{array}{r} .4 \\ \underline{\times .3} \\ .12 \end{array}$$

   a. How many decimal places are there in .4? in .3?

   b. How many decimal places are there in the product numeral?

2. Consider $.03 \times .2 = \square$

$$\begin{aligned} .03 \times .2 &= \frac{3}{100} \times \frac{2}{10} \\ &= \frac{3 \times 2}{100 \times 10} \\ &= \frac{6}{1000} \\ &= .006 \end{aligned}$$

$$\begin{array}{r} .03 \\ \underline{\times .2} \\ .006 \end{array}$$

   a. How many decimal places are there in .03? in .2?

   b. How many decimal places are there in the product numeral?

3. Consider $1.15 \times .05 = \square$

$$\begin{aligned} 1.15 \times .05 &= 1\frac{15}{100} \times \frac{5}{100} \\ &= \frac{115}{100} \times \frac{5}{100} \\ &= \frac{115 \times 5}{100 \times 100} \\ &= \frac{575}{10{,}000} \\ &= .0575 \end{aligned}$$

$$\begin{array}{r} 1.15 \\ \underline{\times .05} \\ .0575 \end{array}$$

a. How many decimal places are there in 1.15? in .05?

b. How many decimal places are there in the product numeral?

4. Examine the following multiplications and complete the table.

		Number of Decimal Places		
		1st factor	2nd factor	Product
a.	3 x .4 = 1.2	0	1	1
b.	.3 x .4 = .12	1	1	2
c.	.03 x .4 = .012			
d.	.03 x .04 = .0012			
e.	.03 x .004 = .00012			

5. Elicit that the number of decimal places in the product numeral is the sum of the number of decimal places in each factor named by a decimal.

6. Compute the following products without changing to fractional form.

   a. .32 x .3 = ☐

      1) We can think of both factors as whole numbers: 32 x 3 = 96. Thus, 96 is part of the product numeral.

      2) How many decimal places must we provide in the product numeral? (2+1 or 3)

      3) Thus, .32 x .3 = .096

      Elicit that a zero is used between the 96 and the decimal point to provide the necessary extra decimal place.

   b. .24 x .15 = ☐

      1) Think of both factors as whole numbers:

$$\begin{array}{r} 15 \\ \underline{\times 24} \\ 60 \\ \underline{30} \\ 360 \end{array}$$

      2) How many decimal places must we provide in the product numeral? (2+2 or 4)

      3) Then .24 x .15 = .0360 = .036.

II. Practice

A. Place the decimal point in the product numeral.

1. $\begin{array}{r} 6.4 \\ \underline{\times\ 3} \\ 192 \end{array}$

2. $\begin{array}{r} 463 \\ \underline{\times .09} \\ 4167 \end{array}$

3. $\begin{array}{r} 34.5 \\ \underline{\times .12} \\ 690 \\ \underline{345} \\ 4140 \end{array}$

B. Find the products.

1. 6 x 1.4
2. 1.5 x 9
3. 1.2 x .6
4. 58 x 3.2
5. 5.8 x 3.2
6. .09 x 2.6
7. 4.2 x 6.18
8. 34.5 x .012
9. .015 x .2
10. .005 x 22.6

C. A motorist averages 17.3 miles on a gallon of gasoline. How many miles can he drive on 21 gallons of gasoline?

D. On a map, the scale is 1 inch = 125 miles. What is the actual distance between two cities which are 4.65 inches apart on the map?

E. At 33.8¢ per gallon, what is the cost of 9.6 gallons of gasoline?

F. A certain type of sheet metal is .027 inches thick. How high is a pile of 250 such sheets?

III. Summary

A. If a whole number is multiplied by a number named by a decimal, how many decimal places will there be in the product numeral?

B. If both factors of a product are named by decimals, explain how you would find the number of decimal places to provide in the product numeral.

Lesson 83

Topic: Division of Rational Numbers: Decimal Form

Aim: To extend understanding of and skill in dividing a rational number named in decimal form by a whole number

Specific Objectives:

Dividing a rational number expressed as a decimal by a whole number
Renaming the dividend to obtain, if possible, a remainder of zero

Challenge: A science class kept records which showed that 4.52 inches of rain fell in a 4-month period.

What was the average monthly rainfall?

I. Procedure

A. Dividing a rational number expressed as a decimal by a whole number

1. Consider $6.3 \div 3 = \square$.

a. We may think of 6.3 as 63 tenths.

1) $$\begin{array}{r} \underline{21 \text{ tenths}} \\ 3\overline{)63 \text{ tenths}} \end{array}$$

2) Thus, $6.3 \div 3 = 2.1$.

b. We may think of $6.3 \div 3$ as $(6 + .3) \div 3$ and use the distributive property of division over addition.

$$\begin{aligned} 6.3 \div 3 &= (6 + .3) \div 3 \\ &= (6 \div 3) + (.3 \div 3) \\ &= 2 + .1 \\ &= 2.1 \end{aligned}$$

Vertical Form

$$\begin{array}{r} 2.1 \\ 3\overline{)6.3} \\ \underline{6.0} \\ .3 \\ \underline{.3} \end{array}$$

c. We may use the fractional form to compute.

$$\begin{aligned} 6.3 \div 3 &= \frac{63}{10} \times \frac{1}{3} \\ &= \frac{63 \times 1}{10 \times 3} \\ &= \frac{63 \times 1}{3 \times 10} \\ &= \frac{63}{3} \times \frac{1}{10} \quad \text{(The quotient } \tfrac{63}{3} \text{ may be found by division.)} \\ &= 21 \times \frac{1}{10} \\ &= 2\frac{1}{10} \text{ or } 2.1 \end{aligned}$$

d. How many decimal places are in the dividend numeral?

e. Elicit that the quotient $6.3 \div 3$ is computed in a way similar to finding the quotient $63 \div 3$.

However, since there is one decimal place in the dividend, there must be one decimal place in the quotient numeral.

2. Refer to the challenge problem.

Elicit that this problem requires the operation of division.

$4.52 \div 4 = \square$

a. Estimate: Since $4.52 > 4$, then $4.52 \div 4 > 4 \div 4$, and $4.52 \div 4 > 1$.

b.

$$\begin{aligned} 4.52 \div 4 &= \frac{452}{100} \times \frac{1}{4} \\ &= \frac{452}{4} \times \frac{1}{100} \\ &= 138 \times \frac{1}{100} \\ &= \frac{138}{100} \text{ or } 1.38 \end{aligned}$$

Vertical Form

$$\begin{array}{r} 1.38 \\ 4\overline{)4.52} \\ \underline{4\ 00} \\ 1\ 52 \\ \underline{1\ 20} \\ 32 \\ \underline{32} \end{array}$$

The average monthly rainfall was 1.38 inches. (Compare with estimate.)

c. How many decimal places are in the dividend numeral? in the quotient numeral?

d. Check by multiplication.

3. After several such examples, have pupils conclude that in dividing a rational number expressed as a decimal by a whole number, we divide as we would with whole numbers. The quotient numeral will have as many decimal places as the numeral for the dividend.

   When using the vertical form for division, the decimal point in the quotient numeral is placed directly above the decimal point in the dividend numeral. This insures the proper number of decimal places in the quotient numeral.

4. Compute the quotients. Check by multiplication.

   a. $6\overline{).18}$

   b. $15\overline{)2.25}$

   c. $4\overline{).921}$

   d. $35\overline{)3.605}$

   e. $18\overline{).2916}$

B. Renaming the dividend

1. Review equivalent decimal names for a number.

   a. What are other decimal names for .5? (.50, .500, .5000, and so on)

   b. Rename as equivalent decimals in hundredths.

      1) .6 2) 1.2 3) 5 4) 26.3

   c. Which of the following is not another name for 6.4?

      1) 6.40 2) 6.04 3) 6.400

   d. Elicit that writing zeros at the end of the dividend numeral does not change the value of the dividend.

2. Consider .5 ÷ 2 = $\square$.

   a. $\begin{array}{r} .2 \\ 2\overline{).5} \\ \underline{4} \\ 1 \end{array}$ or $\begin{array}{r} .2\frac{1}{2} \\ 2\overline{).5} \end{array}$ (Read: 2½ tenths)

   b. Tell pupils that we want to find a quotient named in decimal form and a remainder of zero.

   c. Which decimal name for .5 can be used in computing .5 ÷ 2 so that the remainder will be zero? (.50 or .500 or .5000)

d. $$\begin{array}{r} .25 \\ 2\overline{).50} \\ \underline{40} \\ 10 \\ \underline{10} \end{array} \qquad \begin{array}{r} .250 \\ 2\overline{).500} \\ \underline{400} \\ 100 \\ \underline{100} \end{array} \qquad \begin{array}{r} .2500 \\ 2\overline{).5000} \\ \underline{4000} \\ 1000 \\ \underline{1000} \end{array}$$

e. What is true of $.2\frac{1}{2}$, .25, .250, and .2500?

3. Consider $2.5 \div 4 = \square$.

a. What is the remainder? (.1)

$$\begin{array}{r} .6 \\ 4\overline{)2.5} \\ \underline{2\ 4} \\ 1 \end{array}$$

b. To find the next part of the quotient, how should we rename the dividend?

What is the remainder? (.02)

$$\begin{array}{r} .62 \\ 4\overline{)2.50} \\ \underline{2\ 40} \\ 10 \\ \underline{8} \\ 2 \end{array}$$

c. Continue our work. How should we rename the dividend?

$$\begin{array}{r} .625 \\ 4\overline{)2.500} \\ \underline{2\ 40} \\ 100 \\ \underline{80} \\ 20 \\ \underline{20} \end{array}$$

d. What is the quotient?
What is the remainder?

4. Express the quotient $3 \div 4$ in decimal form with a remainder of zero.

a. Elicit that 3 may be renamed as 3.0, 3.00, and so on.

b. How should we rename 3?
What is the remainder?

$$\begin{array}{r} .7 \\ 4\overline{)3.0} \\ \underline{2\ 8} \\ 2 \end{array}$$

c. Continue our work.
How should we rename the dividend?

$$\begin{array}{r} .75 \\ 4\overline{)3.00} \\ \underline{2\ 8} \\ 20 \\ \underline{20} \end{array}$$

d. What is the quotient?
What is the remainder?

5. Compute the quotients. Rename the dividend, if necessary, so that the remainder is zero.

a. $2\overline{).7}$

b. $4\overline{)1.8}$

c. $5 \div 8$

d. $12\overline{)4.2}$

e. $8\overline{)3.5}$

II. Practice

A. Place the decimal point correctly in the quotient numeral.

1. $5\overline{)45.55}$
2. $3\overline{)18.36}$
3. $6\overline{).036}$
4. $8\overline{).0016}$
5. $24\overline{)12.72}$
6. $17\overline{)59.721}$

B. Compute the quotients in A.

C. Compute the quotients. Rename the dividend, if necessary, so that the remainder is zero.

1. $2\overline{)4.9}$
2. $4\overline{)16.6}$
3. $8.5 \div 10$
4. $11.7 \div 12$
5. $461.8 \div 100$
6. $22\overline{)28.71}$
7. $20\overline{)24.1}$
8. $16\overline{)4.78}$

D. Round each of the quotients in C to the nearest tenth.

E. Mr. Adams used 15 gallons of gasoline to drive 244.5 miles. How many miles per gallon did he average?

F. During one week the Rockville Sports Store sold 28 footballs for $275.80. What was the amount of the average sale?

G. A package of paper measured 1.035 inches in thickness. If there are 345 sheets of paper in the package, what is the average thickness per sheet?

H. On a certain day, 7 inches of snow fell in 4 hours. What was the average snowfall per hour, expressed in hundredths of an inch?

III. Summary

A. When a number named by a decimal is divided by a whole number, how does the number of places in the quotient numeral compare with the number of places in the dividend numeral?

B. In computing a quotient, when would we rename the dividend as an equivalent decimal?

Lessons 84 and 85

Topic: Division of Rational Numbers: Decimal Form

Aim: To learn to divide by a number expressed in decimal form

Specific Objectives:

Reinforcement of multiplication by powers of 10
Review of the use of the property of one for multiplication to change the form of a fraction
Division by a number named in decimal form

Challenge: A piece of board 2 feet in length is to be cut into pieces each .4 feet long. Into how many pieces can the board be cut?

I. Procedure

A. Multiplication by powers of 10

1. Replace the frames.

a. .1 x 10 = □
b. .3 x 10 = □
c. 4.3 x 10 = □
d. .02 x 100 = □
e. .06 x 100 = □
f. 2.17 x 100 = □
g. .001 x 1000 = □
h. .005 x 1000 = □
i. 3.019 x 1000 = □

2. Study the pattern in 1. Use this pattern to find the solutions to the following without using pencil and paper.

a. .4 x 10 = □
b. .26 x 100 = □
c. 10 x 3.8 = □
d. 100 x 2.05 = □
e. .355 x 1000 = □
f. .1168 x 10,000 = □
g. 1000 x .016 = □
h. 10,000 x .138 = □

B. Review of the use of the property of one for multiplication

1. Consider: $2\overline{)8}$

a. What other way can this division be indicated? ($\frac{8}{2}$)

b. When expressing the division in fraction form, what does the numerator correspond to? the denominator?

c. Express in fraction form:

$4\overline{)3}$ $\qquad$ $7\overline{).14}$ $\qquad$ $.2\overline{).8}$ $\qquad$ $.07\overline{).77}$

2. Recall that to find equivalent fractional names for a number, we can multiply by 1 named in fractional form.

a. $\frac{7}{4} \times 1 = \frac{7}{4} \times \frac{2}{2} = \frac{7\times2}{4\times2} = \frac{14}{8}$

b. Elicit that since $\frac{7}{4}$ and $\frac{14}{8}$ are names for the same number, then $7 \div 4 = \Box$ and $14 \div 8 = \Box$ have the same solution.

3. Replace the frames and indicate what form of one you used.

a. $\frac{3}{5} = \frac{\Box}{10}$ $\qquad$ c. $\frac{6}{8} = \frac{24}{\Box}$

b. $\frac{1}{2} = \frac{\Box}{50}$ $\qquad$ d. $\frac{3}{4} = \frac{\Box}{1000}$

4. Without completing each solution, what can you say about the solutions of the sentences in each of the following pairs?

a. $3 \div 5 = \Box$; $6 \div 10 = \Box$ $\qquad$ c. $6 \div 8 = \Box$; $24 \div 32 = \Box$

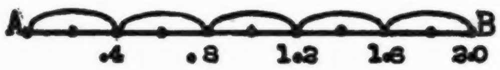
b. $1 \div 2 = \Box$; $25 \div 50 = \Box$ $\qquad$ d. $3 \div 4 = \Box$; $3000 \div 4000 = \Box$

5. **Elicit that when dividend and divisor are multiplied by the same non-zero number, the resulting division has the same quotient as the original division.**

C. Division by a number named in decimal form

1. Refer to the challenge problem. Elicit that to solve this problem, we use the operation of division. $2 \div .4 = \Box$

Pupils may suggest using the number line to find the solution to the problem. The measure of $\overline{AB}$ in inches is 2.



a. Into how many equal segments each measuring .4 inches has $\overline{AB}$ been partitioned?

b. $2 \div .4 = 5$ The board can be cut into 5 pieces.

2. Elicit that the use of a number line is not always convenient. We would therefore like a method for dividing by a number named in decimal form which is similar to dividing by a whole number.

3. Have pupils see that we can use the property of one for multiplication to name any fractional number divisor as a whole number.

   a. $2 \div .4 = \frac{2}{.4} = \frac{2}{.4} \times 1 = \frac{2}{.4} \times \frac{10}{10} = \frac{20}{4}$

      $20 \div 4 = 5$ $\qquad\qquad$ $2 \div .4 = 5$

      Why did we rename 1 as $\frac{10}{10}$?

   b. $15 \div .05 = \frac{15}{.05} = \frac{15}{.05} \times 1 = \frac{15}{.05} \times \frac{100}{100} = \frac{1500}{5}$

      $1500 \div 5 = 300$ $\qquad\qquad$ $15 \div .05 = 300$

      Why did we rename 1 as $\frac{100}{100}$?

   c. $2 \div .004 = \frac{2}{.004} = \frac{2}{.004} \times 1 = \frac{2}{.004} \times \frac{1000}{1000} = \frac{2000}{4}$

      $2000 \div 4 = 500$ $\qquad\qquad$ $2 \div .004 = 500$

      Why did we rename 1 as $\frac{1000}{1000}$?

4. Consider $2.8\overline{)23.24}$.

   a. What name for 1 would you use to rename the divisor of $\frac{23.24}{2.8}$ as a whole number?

   b. Instead of saying that we multiplied $\frac{23.24}{2.8}$ by $\frac{10}{10}$, we can say that we multiplied the divisor and the dividend by 10. We can show this as $28\overline{)232.4}$.

   c. What is true of $2.8\overline{)23.24}$ and $28\overline{)232.4}$? (they have the same quotient)

   d. $$\begin{array}{r} 8.3 \\ 28\overline{)232.4} \\ \underline{224} \\ 8\,4 \\ \underline{8\,4} \end{array} \qquad\qquad \begin{array}{r} \underline{\text{Check}} \\ 8.3 \\ \underline{\times 2.8} \\ 66\,4 \\ \underline{166} \\ 23.24 \end{array}$$

5. After several similar examples have pupils conclude that if the divisor is named in decimal form, we change it to a whole number by multiplying by the appropriate power of 10. The dividend must be multiplied by the same power of ten.

II. Practice

A. Which of the following name the same quotient? Explain.

1. $.27\overline{)9.18}$
2. $27\overline{)918}$
3. $.27\overline{)91.8}$
4. $2.7\overline{)91.8}$

B. By which power of 10 would you multiply both divisor and dividend in each of the following in order to have a whole number divisor?

1. $9 \div .3$
2. $16 \div .8$
3. $1.5 \div .05$
4. $.75 \div .15$
5. $6.3 \div .007$
6. $25.6 \div .32$
7. $69 \div 1.15$
8. $.048 \div .012$
9. $15.2\overline{)65.36}$
10. $2.24\overline{)3.584}$

C. Mr. Adams drove 246.5 miles and found that he had used 14.5 gallons of gasoline. How many miles per gallon did he average?

D. A school bookstore sold notebooks at 29¢ each. How many notebooks were sold if the cashier took in $22.62 on the sale of notebooks?

E. A package of paper measures 1.575 inches in thickness. If each sheet of paper is .005 inches thick, how many sheets of paper are in the package?

III. Summary

A. Elicit steps in dividing by a number named in decimal form.

1. Change the divisor to a whole number by multiplying by a power of ten.
2. Multiply the dividend by the same power of ten.
3. Rewrite the example with the new divisor and new dividend.
4. Divide

B. How do we decide by which power of ten we will multiply both divisor and dividend?

C. When we multiply both divisor and dividend by the same power of ten, what number property are we making use of?

CHAPTER IX

The understanding that the development of units of measure depends upon the aspect of what is to be measured is given important emphasis in this chapter. Teaching procedures are suggested for developing and extending the following concepts and skills:

meaning of linear measure
facility in the use of a ruler
the approximate nature of measurement
metric system of measurement: linear, weight, liquid
measurement of the perimeter of a polygon
measurement of the circumference of a circle
meaning of square measure
measurement of the area of a rectangular region; of a square region
angle measurement and some special angle relationships

Although most pupils know something about standard units of measure, they may not have thought about how units of measure must be of the same kind as the entity we wish to measure. Whereas a small line segment is used as a unit of linear measure, a small area is used as a unit of area measure, a small unit angle is used to measure angles, and so on.

One of the most important concepts in the topic of measurement is the concept of the approximate nature of a measurement as opposed to the exactness of counting. As pupils gain facility in the use of a ruler, they develop the understanding that errors are inevitable in measurement. We cannot avoid these errors, but we can keep them within certain limits. Pupils are guided to realize that the smaller the subdivision on the measuring instrument, the greater the precision of the measurement. Thus, a measurement of 3 inches is more precise than one of 3 feet; a measurement of 3 feet is more precise than one of 3 yards.

In working with metric measures, pupils should realize the relative ease of changing from one unit to another within the metric system. This is due to the fact that each metric unit is a decimal part, or decimal power of every other unit.

Whereas closed geometric figures are measurable in terms of units of length, the regions formed by the union of the figures and their interiors are measurable in terms of units of area. The traditional approach to the concept of area did not always make clear the distinction between a simple closed curve and its interior. The interior is not part of the figure itself. It is that portion of the plane which is enclosed by the figure. The word "area" therefore should be used to stand for the measure of a region, that is, the measure of the union of the polygon with its interior.

Pupils have previously discussed the separation of the plane by an angle into three sets of points: the set of points inside the figure; the set of points outside the figure; the set of points on the figure. The portions of the plane both inside and outside an angle are not closed and, therefore, a unit of measure such as the square inch is inappropriate for the measurement of these regions. Pupils should be guided to realize that just as we use a unit length to measure length, and a unit square to measure surface, so we use a unit angle to measure an angle. A standard unit of angle measure is the degree.

The newer approaches to the teaching of angle measurement emphasize the distinction between an angle and the measure of an angle. An angle is a set of points. The measure of an angle is a number. For example, an angle, A, may have a degree measure of 23. We report the size of this angle as 23°. In symbols, $m\angle A = 23^\circ$. Some textbooks use the symbols $m\angle A = 23$, and some, $\angle A = 23^\circ$.

In developing pupil skill in using a protractor, it is recommended that each pupil have his own protractor. The classroom also should be equipped with several protractors to be used at the chalkboard. Pupils should understand that of the two scales usually found on a protractor, they may use whichever scale is more convenient.

In addition to having pupils learn the classification of angles as acute, right, obtuse, and straight, the concepts of supplementary, complementary and vertical angles are presented. The understanding is then developed that if two angles are supplementary (complementary) and the measure of one of them is known, the measure of the other may be determined immediately without using any measuring instrument.

CHAPTER IX

MEASUREMENT

Lessons 86-106

Lesson 86

Topic: Linear Measure

Aim: To reinforce and extend understanding of linear measure

Specific Objectives:

Concept of unit of linear measure
Measuring a line segment; the notation m ($\overline{AB}$)
Congruence of line segments

Challenge: Measure the length of your desk in handspans and then in inches. Which unit of measure is more satisfactory? Why?

I. Procedure

A. Concept of unit of linear measure

1. In discussing the challenge question with pupils, elicit that non-standard units of measure, such as handspans, digits, and cubits are not satisfactory because these vary from person to person.

2. Name some standard units of length used in measuring. (inch, foot, yard, mile) Why are these called standard units?

3. Why do we not use the pound, the hour, or the gallon in measuring length? (We must use a unit of length to measure length.)

4. Elicit that measurement of length is the process of comparing an unknown length (such as the length of the desk) with a basic unit of length, non-standard or standard.

B. Measuring a line segment

1. Distribute a rexographed sheet of paper with line segments such as the following:

Figure 1

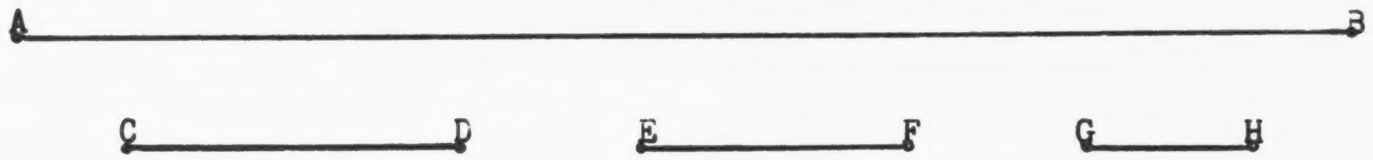
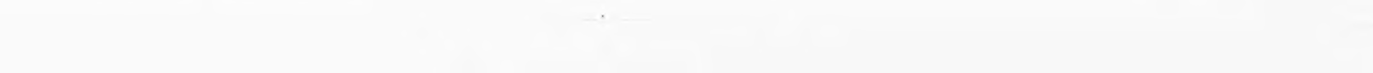


Figure 2



a. Refer to Figure 1. Measure $\overline{AB}$ using $\overline{CD}$ as the unit of measure.

   Note: This may be done by setting the opening of a compass to match $\overline{CD}$, or by placing the edge of a sheet of paper along $\overline{CD}$ and marking points C and D on the paper so that they match the points on the diagram.

   1) Elicit that a segment the length of $\overline{CD}$ is contained in $\overline{AB}$ approximately 4 times.

   2) Tell pupils that we say the measure of $\overline{AB}$ is approximately equal to 4 times the measure of $\overline{CD}$. The unit of measure is $\overline{AB}$ and the number 4 is the measure of AB when AB is measured in units of $\overline{CD}$.

b. Measure $\overline{AB}$ using $\overline{EF}$ as the unit of measure.

   1) Elicit that a segment the length of $\overline{EF}$ is contained in $\overline{AB}$ approximately 5 times.

   2) What is the measure of $\overline{AB}$ when it is measured in units of $\overline{EF}$?(5)

c. Measure $\overline{AB}$ using $\overline{GH}$ as the unit of measure. What is the measure of $\overline{AB}$ when it is measured in units of $\overline{GH}$? (8)

d. Why does the measure of $\overline{AB}$ vary? Elicit that the measure of a line segment varies according to the unit of measure being used. It is therefore necessary to know not only the number of times the unit segment is contained in the measured segment, but also the size of the unit used.

2. Have pupils use their rulers to measure $\overline{AB}$. They find that $\overline{AB}$ is approximately six inches long.

   a. What is the unit of measure? (an inch)

   b. What is the measure? (6)
   We write: $m\ (\overline{AB}) \approx 6$ (The symbol "$\approx$" is used to mean "approximately equal to")

   c. What is the length? (6 inches, since the length of a segment includes both the measure and the unit of measure)

   d. We may use the symbol $\overline{AB}$ without the bar above it to mean the length of $\overline{AB}$. Thus, to say the length of $\overline{AB}$ is approximately 6 inches, we write: AB $\approx$ 6 inches.

3. Refer to Figure 2. Use the inch as the unit of measure and replace the frames.

   a. $m(\overline{RX}) = \square$

   b. $m(\overline{RY}) = \square$

   c. $m(\overline{NS}) = \square$

   d. $m(\overline{YO}) = \square$

   e. $m(\overline{RS}) = \square$

C. Congruence of line segments

1. Consider the line segments pictured in Figure 2.

   a. Name the segments which measure 1 inch.
   ($\overline{RX}$, $\overline{XY}$, $\overline{YM}$, $\overline{MN}$, $\overline{NO}$, $\overline{OS}$)

   b. Name the segments which have a length of 2 inches; 3 inches; 4 inches.

   c. How many segments have a length of 5 inches? Name them.
   ($\overline{RO}$ and $\overline{XS}$)

   d. Since $\overline{RO}$ and $\overline{XS}$ are each 5 inches long, we can write:
   $m(\overline{RO}) = m(\overline{XS})$ or $RO = XS$.

   e. Tell pupils that when two line segments have the same measure, they are said to be congruent.

   $\overline{RO}$ is congruent to $\overline{XS}$. Symbolically, $\overline{RO} \cong \overline{XS}$.

2. Refer to Figure 2.

   a. Which line segments are congruent to $\overline{RX}$? Explain.

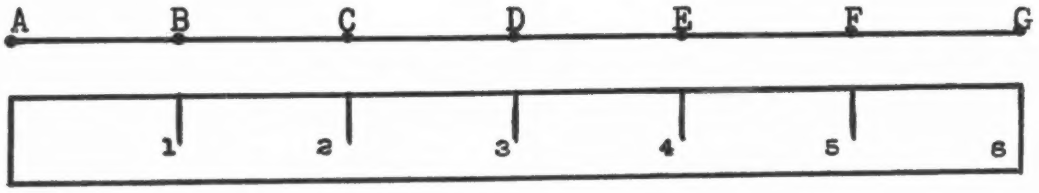
   b. Which line segments are congruent to $\overline{RN}$? Explain.

II. Practice

A. Non-standard units of measure such as a pinch of salt are still being used today. Can you think of any others?

B. What are some standard units used in measuring time? in measuring weight?

C. Refer to the figure below:



1. Replace the frames.

a. $m(\overline{AB}) = \square$　　c. $m(\overline{CF}) = \square$　　e. $m(\overline{CG}) = \square$

b. $m(\overline{AD}) = \square$　　i. $m(\overline{BG}) = \square$　　f. $m(\overline{AF}) = \square$

2. Replace the frames.

a. $m(\square) = 3$　　c. $m(\square) = 1$

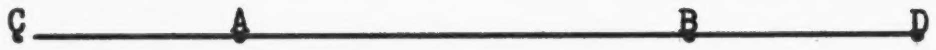
b. $m(\square) = 6$　　d. $m(\square) = 4$

3. Label each of the following sentences true or false.

a. $\overline{BC}$ is congruent to $\overline{FG}$　　c. $\overline{AG}$ is congruent to $\overline{BF}$

b. $\overline{BD}$ is congruent to $\overline{FG}$　　d. $\overline{AC}$ is congruent to $\overline{EG}$

4. Draw a picture of a line segment which measures 4 inches. Name the endpoints C and D. Mark a point on $\overline{CD}$ that is 1 inch from C. Mark it A. Mark a point on $\overline{CD}$ that is 1 inch from D. Mark it B.



a. What is $m(\overline{CB})$? What is CB?

b. What is $m(\overline{BD})$? What is BD?

c. Name two pairs of congruent line segments.

III. Summary

A. In measuring the length of an object, what are we comparing?

B. Why is it possible for the measure of a line segment to vary?

C. Name some standard units of linear measure.

D. What is meant by the symbol $\overline{PQ}$? $m(\overline{PQ})$

E. If the measure of two (or more) line segments is the same, how can we describe the two line segments?

F. What new vocabulary did you learn today?
(congruent)

G. What new symbols did you learn today?

Lesson 87

Topic: Linear Measure

Aim: To reinforce and extend understanding of the approximate nature of measurement

Specific Objectives:

Precision in measurement
Greatest possible error of measurement

Challenge: When we measure a line segment very carefully and decide that it is 2¼ inches long, can we say that the length of the segment is exactly 2¼ inches?

I. Procedure

A. Precision in measurement

1. Distribute a rexographed sheet of paper with drawings of line segments and scales such as the following:

Figure 1

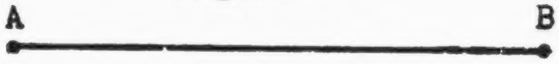


Figure 2

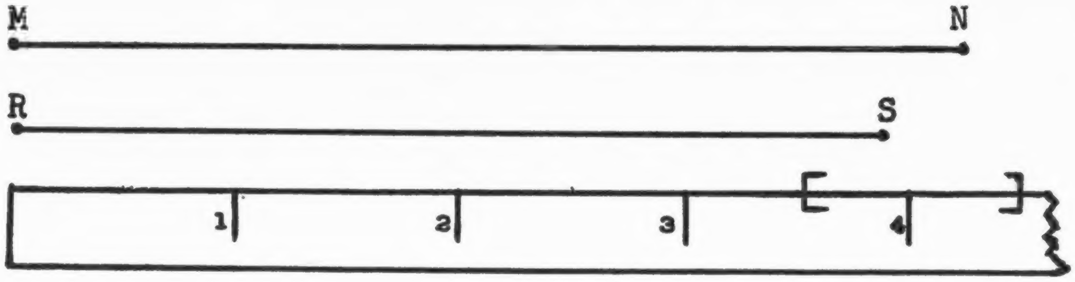


Figure 3

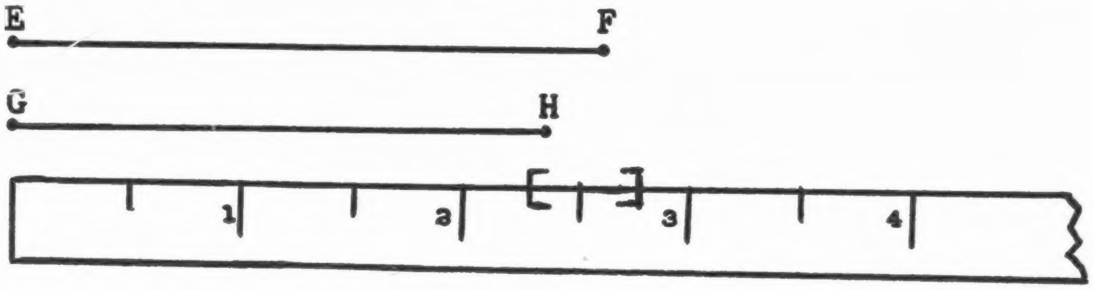
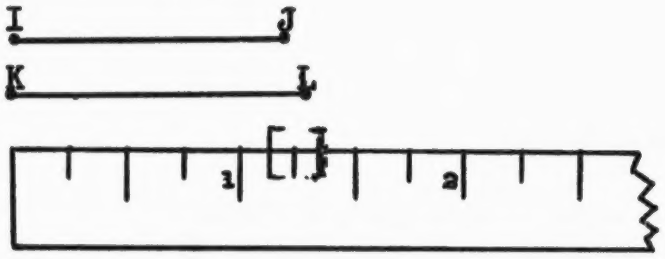


Figure 4



a. Consider the line segment in Figure 1. Suppose the unit of measure is the inch and the ruler you are using is marked off in inches.

   1) What is the length of $\overline{AB}$ to the nearest inch? (2 inches)

   2) Have pupils recall that measurement with a ruler marked off in inches is said to be "correct" to the nearest inch.

b. Suppose the unit of measure is $\frac{1}{2}$ inch and your ruler is marked off in $\frac{1}{2}$ inches.

   1) What is the length of $\overline{AB}$ to the nearest $\frac{1}{2}$ inch? ($\frac{5}{2}$ inches)

   2) How precise are measurements with a ruler marked off in $\frac{1}{2}$ inches?

c. Suppose the unit of measure is $\frac{1}{4}$ inch and your ruler is marked off in $\frac{1}{4}$ inches.

   1) What will be the precision of a measurement with this ruler?

   2) What is the length of $\overline{AB}$ to the nearest $\frac{1}{4}$ of an inch? ($\frac{9}{4}$ inches)

2. Which of the measurements of $\overline{AB}$ is correct? Elicit that they are all correct but some of the measurements are more precise than others. The smaller the unit of measure used, the more precise the measurement will be.

3. Refer to the challenge question. Have pupils see that no measurement can ever be "perfect" or exact. Even if we were to use extremely small units or subdivisions on our scale,

a "tiny part" of what we are measuring would always extend beyond some mark on the scale. We therefore say that every measurement is approximate.

4. Have pupils measure lengths of various objects and line segments to different degrees of precision.

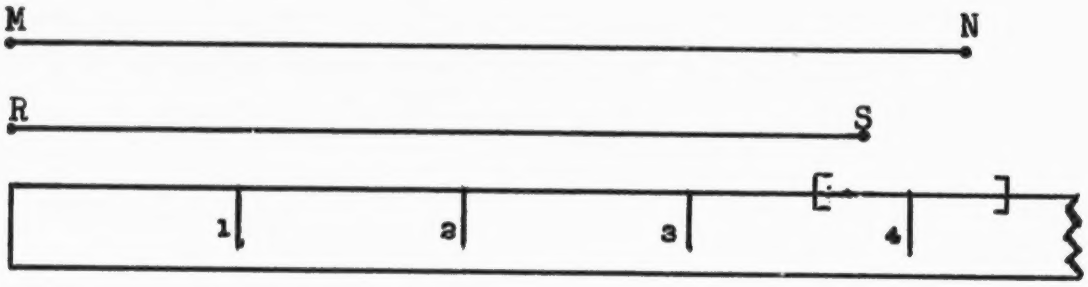
   ($\frac{1}{2}$'s $\frac{1}{4}$'s $\frac{1}{8}$'s $\frac{1}{16}$'s)

   Have them see that they show the precision of the measurement by the denominator of the fraction they use. For example, they would write $2\frac{2}{4}$ inches, not $2\frac{1}{2}$, if the unit of measure is $\frac{1}{4}$ inch; $3\frac{8}{16}$ inches, not $3\frac{1}{2}$, if the unit of measure is $\frac{1}{16}$ inch.

B. Greatest possible error of measurement

1. Discuss with pupils the fact that since no measurement is ever "perfect" or exact, there is a difference between the actual length of an object and the measured length. We call this difference the error of the measurement. With any particular scale we may use, how large will the error be?

2. Have pupils examine the picture of the line segments and the scale in Figure 2 of the rexographed sheet.



   a. What is the length of $\overline{RS}$ to the nearest inch? (4 inches)

   b. What is the length of $\overline{MN}$ to the nearest inch? (4 inches)

   c. Have pupils observe that the actual lengths of $\overline{RS}$ and of $\overline{MN}$ are between $3\frac{1}{2}$ inches and $4\frac{1}{2}$ inches. Each length is not more than $\frac{1}{2}$ inch from 4 inches.

   d. Elicit that the greatest possible error that could be made in measuring with an inch scale is $\frac{1}{2}$ inch.

3. Consider the picture of the line segments and the scale in Figure 3 of the rexographed sheet.

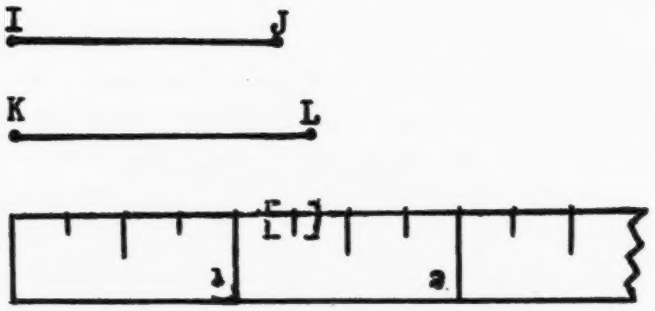
a. What is the length of $\overline{EF}$ to the nearest $\frac{1}{2}$ inch? (5 half-inches of $\frac{5}{2}$ inches)

b. What is the length of $\overline{GH}$ to the nearest $\frac{1}{2}$ inch? ($\frac{5}{2}$ inches)

c. Have pupils observe that the actual lengths of $\overline{EF}$ and $\overline{GH}$ are between $\frac{9}{4}$ inches and $\frac{11}{4}$ inches. Each length is within $\frac{1}{4}$ inch of $\frac{5}{2}$ inches.

d. Elicit that the greatest possible error that could be made with a half-inch scale is $\frac{1}{4}$ inch, that is to say, half of $\frac{1}{2}$ inch.

4. Consider the picture of the line segments and the scale in Figure 4 of the rexographed sheet.



a. What is the length of $\overline{IJ}$ to the nearest $\frac{1}{4}$ inch? ($1\frac{1}{4}$ inches)

b. What is the length of $\overline{KL}$ to the nearest $\frac{1}{4}$ inch? ($1\frac{1}{4}$ inches)

c. Elicit that the actual lengths of $\overline{IJ}$ and $\overline{KL}$ are between $1\frac{1}{8}$ and $1\frac{3}{8}$ inches. Each length is not more than $\frac{1}{8}$ of an inch from $1\frac{1}{4}$.

d. What is the greatest possible error that could be made with a quarter-inch scale? ($\frac{1}{8}$ inch)

5. Elicit that the error we make in using a measuring instrument can never be greater than one-half of the smallest unit of measure on the scale of the instrument. Tell pupils that this error is called the greatest possible error of the measurement.

6. What is the greatest possible error of measurement if the smallest unit of measure is:

a. $\frac{1}{4}$ foot

b. $\frac{2}{10}$ inch

c. $\frac{1}{8}$ yard

d. $\frac{6}{8}$ inch

e. $\frac{1}{16}$ mile

II. Practice

A. Draw pictures of 3 line segments of varying lengths

1. Measure these line segments using $\frac{1}{16}$ of an inch as the unit of measure.

2. Measure these line segments again, but use $\frac{1}{2}$ of an inch as the unit of measure.

B. Which of these measurements is most precise? Explain.

1. 7 inches
2. 3 feet
3. $5\frac{1}{4}$ inches
4. $4\frac{1}{4}$ yards
5. $1\frac{7}{8}$ inches

C. Which of these measurements is most precise? Explain.

1. $1\frac{1}{2}$ inches
2. $1\frac{2}{4}$ inches
3. $1\frac{4}{8}$ inches
4. $1\frac{8}{16}$ inches

D. For each measurement in B, between which two measurements does the actual measurement lie? (7 inches: between $6\frac{1}{2}$ inches and $7\frac{1}{2}$ inches, and so on)

E. Complete the table.

	Measurement	Precision of Measurement	Greatest Possible Error
1.	$3\frac{2}{4}$ inches	nearest $\frac{1}{4}$ inch	$\frac{1}{8}$ inch
2.	$2\frac{4}{10}$ feet		
3.	$7\frac{1}{2}$ yards		
4.	$1\frac{15}{16}$ inches		
5.	$2\frac{3}{16}$ inches		

F. Perform the indicated operation.

1. 8 feet 3 inches + 12 feet 9 inches
2. 3 feet 6 inches - 2 feet 5 inches
3. 15 yards 1 foot - 10 yards 2 feet
4. 3 yards 2 feet 6 inches + 1 yard 1 foot 8 inches
5. 8 yards 24 inches - 6 yards 30 inches

G. Perform the indicated operation.

1. 2 x (3 feet 5 inches)
2. 10 x (8 yards 1 foot)
3. 6 x (6 yards 12 inches)
4. 8 feet 6 inches ÷ 2
5. 6 yards 2 feet ÷ 5

III. Summary

A. Why do we say that all measurement is approximate?

B. What is meant by the precision of a measurement?

C. What is meant by the error of a measurement?

D. What is meant by the greatest possible error of measurement?

E. What mathematical vocabulary have we used today?

(precision, greatest possible error)

Lessons 88 and 89

Topic: The Metric System of Measurement

Aim: To understand the decimal nature of the metric system of linear measurement

Specific Objectives:

Relationships among the linear units of the metric system
The decimal nature of the metric system
Converting from one unit to another in the metric system

Supplementary Teaching Aids: A class set of rulers marked in English and in metric units; several meter sticks; a yardstick.

Motivation: A popular size of home movie film is 8 mm (millimeter). What is the meaning of a millimeter?

I. Procedure

A. Relationships among the linear units of the metric system

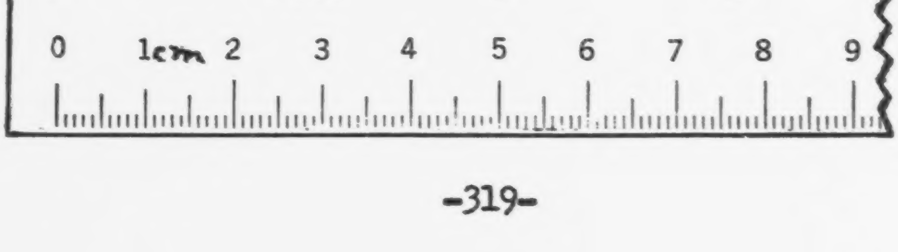
1. Elicit additional examples of the use of millimeter measurement that pupils may know: photography, scientific equipment, and so on.

2. Conclude that this unit belongs to a system of measurement different from our own. Tell pupils that this system is called the metric system of measurement

3. Elicit that every system of measurement must have a basic standard unit. The meter is the basic unit of length in the metric system.

4. Display a meter stick. Have pupils compare its length with that of a yardstick. They observe the meter is somewhat longer than the yard.

5. Have pupils study the metric ruler.



a. The smallest subdivisions of the edge marked in English units represent congruent segments measuring $\frac{1}{16}$ of an inch.

The smallest subdivisions of the edge marked in metric units represent congruent segments measuring 1 millimeter (1 mm).

b. There are 10 millimeter divisions to each larger numbered division called a centimeter (cm). 10 mm = 1 cm

c. Have pupils use the metric scale to measure the length of items, such as pencils or textbooks, in centimeters, in millimeters. Have them make observations about objects in the room: "My pencil is about 1 centimeter thick." "A penny is about 20 millimeters across."

6. Distribute meter sticks.

a. Have pupils note that the markings on the meter stick are set off in groups of 10 (10, 20, 30, ..., 100).

b. Have them apply their metric rulers to the meter sticks and observe that 10 denotes 10 cm; 20 denotes 20 cm; and so on.

c. Tell pupils that each group of 10 centimeters equals 1 decimeter (dm). 10 cm = 1 dm

d. Elicit further that 10 decimeters equal 1 meter. 10 dm = 1 m

7. Summarize the metric equivalents

10 mm = 1 cm or 1 mm = $\frac{1}{10}$ cm (.1 cm)

10 cm = 1 dm or 1 cm = $\frac{1}{10}$ dm (.1 dm)

10 dm = 1 m or 1 dm = $\frac{1}{10}$ m (.1 m)

8. Discuss the motivation question.

B. The decimal nature of the metric system

1. Have pupils relate the metric units to the basic unit, the meter.

a. Meaning of prefixes

1) deci means a tenth (decimal)
2) centi means a hundredth (cent)
3) milli means a thousandth (a real estate tax is sometimes expressed in mills)

b. The value of the units in terms of the basic unit, the meter.

1) How many decimeters are there in a meter?
   What part of the meter is the decimeter? ($\frac{1}{10}$ or .1 of a meter)
2) How many centimeters are there in a meter?
   What part of the meter is the centimeter? ($\frac{1}{100}$ or .01 of a meter)
3) How many millimeters are in a meter?
   What part of the meter is the millimeter? ($\frac{1}{1000}$ or .001 of a meter)

c. Show these metric equivalents in table form as follows:

1 mm = .001 m or $\frac{1}{1000}$ m
1 cm = .01 m or $\frac{1}{100}$ m
1 dm = .1 m or $\frac{1}{10}$ m
1 m = 1. m

1) How many times as large as the millimeter is the centimeter? the decimeter as the centimeter? the meter as the decimeter?
2) How many times as large as the millimeter is the decimeter?

d. Follow the pattern in c and replace the frames.

1 dekameter (dkm) = □ m (10)
1 hectometer (hm) = □ m (100)
1 kilometer (km) = □ m (1000)

1) How many times as large as a meter is the dekameter? the hectometer as the dekameter? the kilometer as the hectometer?

2) How many times as large as a dekameter is a kilometer?

e. How many times as large as a metric unit is the next larger metric unit?

How many times as large as a metric unit is the next smaller metric unit?

f. Have pupils realize that the metric system, like our numeration system, is a decimal system. That is to say, it is a system based on powers of 10.

C. Converting from one unit to another unit within the metric system

1. Review multiplication by 10, 100, 1000, .1, .01, .001.

2. Have pupils draw a picture of a line segment 3.5 centimeters in length.

a. Use your metric ruler to find the measure of the line segment in millimeters. (35 mm)

b. How can we convert 3.5 centimeters to millimeters without using a metric scale?

3. Have pupils convert one metric unit to another as follows:

a. 3 mm = 3 x 1 mm
= 3 x .1 cm (since 1 mm = .1 cm)
= .3 cm

b. 52 mm = 52 x 1 mm
= 52 x .1 cm (why?)
= 5.2 cm

c. 3.5 cm = 3.5 x 1 cm
= 3.5 x 10 mm (since 1 cm = 10 mm)
= 35 mm

d. 3.5 cm = 3.5 x 1 cm
= 3.5 x .01 m (since 1 cm = .01 m)
= .035 m

e. 2m 5cm = 2m + 5cm
= 2m + .05m
= 2.05m

4. Have pupils see that converting from one metric unit of length to another involves only multiplication by 10, or 100, or 1000, or .1, and so on. This computation is far simpler than that needed to change from one English unit to another.

II. Practice

A. Use a metric scale and measure the following objects in centimeters and millimeters.

1. the length of your pencil
2. the thickness of your mathematics book
3. the width of a page in your notebook

B. Replace the frames.

1. □ mm = 1 cm
2. □ cm = 1 m
3. □ mm = 1 m
4. □ m = 1 km
5. □ cm = 1 dm
6. □ dm = 1 km

C. Tell how you would change

1. 120 yards to feet
2. 8 feet to inches
3. 400 centimeters to meters
4. 120 centimeters to millimeters
5. 5 kilometers to meters

D. Complete the table

	mm	cm	m
1.	?	?	1
2.	8	?	?
3.	?	.6	?
4.	?	?	3
5.	?	15	?

E. Convert these measurements as indicated

1. 72 centimeters to millimeters
2. 9 kilometers to meters
3. 615 centimeters to meters
4. 35 meters to centimeters
5. 16 decimeters to millimeters
6. 6500 meters to kilometers

F. Draw a picture of line segment XY which is 25 mm in length; then draw a picture of a line segment KL which is 2 cm 5 mm in length.

Why are the two line segments congruent?

G. Draw pictures of three line segments of different lengths. Find the measure of each to the nearest centimeter; to the nearest millimeter.

H. Replace the frames.

1. 1 meter 56 decimeters = □ meters
2. 16 centimeters 8 millimeters = □ centimeters
3. 16 centimeters 8 millimeters = □ millimeters
4. 8.5 centimeters = □ millimeters
5. 10.8 kilometers = □ meters
6. 689 meters = □ kilometers

III. Summary

A. What is the basic unit of measure in the metric system of linear measurement?

B. Name four units of linear metric measure and tell what relationship each has to the meter.

C. What is the relationship of any metric unit to the next smaller unit? to the next larger unit?

D. How is the metric system similar to our numeration system? (they are both decimal systems)

E. Why is it easier to convert from one metric unit to another than it is to convert from one English unit of linear measure to another?

F. What new vocabulary have you learned today?

(meter, millimeter, centimeter, decimeter, dekameter, hectometer, kilometer)

Lesson 90

Topic: The Metric System of Measurement

Aim: To learn to compute with metric measures

Specific Objectives:

Addition and subtraction of metric measures
Multiplication and division with metric measures

Challenge: In performing an experiment, a scientist observed that an object first moved a distance of 8 cm 5 mm and then moved an additional distance of 7 cm 6 mm. What was the total distance moved by the object?

I. Procedure

A. Addition and subtraction of metric measures

1. Review procedures for addition and subtraction with English linear measures, using feet and inches.

2. Elicit that the challenge problem requires the operation of addition.

a. 8 cm 5 mm + 7 cm 6 mm = □

b.
$$\begin{aligned} 8\text{ cm }5\text{ mm} + 7\text{ cm }6\text{ mm} &= (8\text{ cm} + 7\text{ cm})+(5\text{ mm} + 6\text{ mm}) \\ &= 15\text{ cm} + 11\text{ mm} \\ &= 16\text{ cm} + 1\text{ mm (since }10\text{ mm} = 1\text{ cm)} \\ &= 16\text{ cm }1\text{ mm} \end{aligned}$$

c. Vertical form:

$$\begin{array}{r} 8\text{ cm }5\text{ mm} \\ \underline{+7\text{ cm }6\text{ mm}} \\ 16\text{ cm }1\text{ mm} \end{array} \quad \text{or} \quad \begin{array}{r} 8.5\text{ cm} \\ \underline{+7.6\text{ cm}} \\ 16.1\text{ cm} \end{array}$$

d. After several such examples, elicit that because the metric system is based on ten, adding measures in the metric system is similar to adding whole numbers or fractional numbers named in decimal form.

3. Refer again to the challenge problem.
   How much greater than the first distance moved by the object is the second distance?

   a. 8 cm 5 mm - 7 cm 6 mm = $\square$

   b. $$\begin{array}{rcr} 8\text{ cm }5\text{ mm} &=& 7\text{ cm }15\text{ mm} \\ \underline{-7\text{ cm }6\text{ mm}} &=& \underline{7\text{ cm }\;\;6\text{ mm}} \\ && 9\text{ mm} \end{array} \quad\text{or}\quad \begin{array}{r} 8.5\text{ cm} \\ \underline{-7.6\text{ cm}} \\ .9\text{ cm} \end{array}$$

   Why was it necessary to rename 8 cm 5 mm as 7 cm 15 mm?

   c. After several such examples elicit that subtracting measures in the metric system is similar to subtracting whole numbers or fractional numbers named in decimal form.

B. Multiplication and division with metric measures

1. Review procedures for multiplication and division with English linear measures.

2. Pose problem: Louis has two pieces of wire. In metric measure, the shorter piece of wire is 1 m 60 cm long. If the other piece of wire is twice as long, what is the length of the longer piece?

   a. 2 × 1 m 60 cm = $\square$

   b. $$\begin{aligned} 2 \times 1\text{ m }60\text{ cm} &= 2 \times (1\text{ m} + 60\text{ cm}) \\ &= 2 \times (1\text{ m}) + (2 \times 60\text{ cm}) \\ &= 2\text{ m} + 120\text{ cm} \\ &= 3\text{ m }20\text{ cm} \end{aligned}$$

   c. Vertical form:

   $$\begin{array}{c} 1\text{ m }60\text{ cm} \\ \underline{\times\ 2\qquad\qquad\qquad\qquad} \\ 2\text{ m }120\text{ cm} = 3\text{ m }20\text{ cm} \end{array} \quad\text{or}\quad \begin{array}{r} 1.60 \\ \underline{\times\ 2\ \ } \\ 3.20\text{ m} \end{array}$$

   d. Elicit again that because the metric system is a decimal system, multiplication with metric measures is similar to multiplication with whole numbers or fractional numbers named in decimal form.

3. Refer again to problem posed in B-2.
   Louis needs half of the shorter piece of wire to hang a picture. What is the length of the wire that he needs for the picture?

   a. Elicit that we may use division to solve the problem.
      1 m 60 cm ÷ 2 = $\square$

b. 1 m 60 cm ÷ 2 = 160 cm ÷ 2
= 80 cm

Why was it necessary to rename 1 m 60 cm as 160 cm?

c. Vertical form:

$$\begin{array}{r} 80\text{ cm} \\ 2\overline{)\cancel{1\text{ m}}\ \overset{160}{\cancel{60}}\text{ cm}} \end{array} \qquad \text{or} \qquad \begin{array}{r} .80\text{ m} \\ 2\overline{)1.60} \end{array}$$

d. Why is division with metric measures similar to division with whole numbers or fractional numbers named in decimal form?

II. Practice

A. Replace the frames.

1. 5 cm 18 mm = □ cm 8 mm
2. 3 m 190 cm = 4 m □ cm
3. 9 km 1500 m = □ km 500 m
4. 5 cm 8 mm = 4 cm □ mm
5. 13 cm 12 mm = 12 cm □ mm
6. 8 m 6 cm = □ m 106 cm
7. 12 km 30 m = □ km 1030 m

B. Replace the frames.

12 cm 5 mm + 8 cm 9 mm = (12 cm + □) + (8 cm + Δ)
= (12 cm + ∇) + (5 mm + 9 mm)
= ○ cm + ◇ mm
= ▭

C. Perform the indicated operation.

$$\begin{array}{rr} 1. & 35\text{ cm }3\text{ mm} \\ & \underline{+16\text{ cm }7\text{ mm}} \end{array}$$

$$\begin{array}{rr} 2. & 6\text{ m }80\text{ cm} \\ & \underline{+7\text{ m }40\text{ cm}} \end{array}$$

$$\begin{array}{rr} 3. & 6\text{ m }20\text{ cm }5\text{ mm} \\ & \underline{+3\text{ m }80\text{ cm }5\text{ mm}} \end{array}$$

$$\begin{array}{rr} 4. & 16\text{ cm }2\text{ mm} \\ & \underline{-10\text{ cm }4\text{ mm}} \end{array}$$

$$\begin{array}{rr} 5. & 56\text{ m }6\text{ cm} \\ & \underline{-29\text{ m }9\text{ cm}} \end{array}$$

$$\begin{array}{rr} 6. & 26\text{ km }42\text{ m} \\ & \underline{-18\text{ km }56\text{ m}} \end{array}$$

D. Perform the indicated operation.

1. 8 x 5 cm 3 mm
2. 10 x 6 m 32 cm
3. 13 x 9 km 600 m
4. 9 cm 6 mm ÷ 3
5. 16 cm 35 mm ÷ 5
6. 37 km 40 m ÷ 8

E. Bob's cousin, who lives in France, wrote that the distance from his home to his school is 2 kilometers 560 meters. What is the round trip distance?

F. For a science demonstration, a piece of wire 13 centimeters 5 millimeters long was cut into 3 pieces of equal length. What is the length of each piece?

III. Summary

A. In what way is computation with metric measures similar to computation with numbers?

B. How is computation with metric linear measures similar to computation with English linear measures?

C. How is computation with metric linear measures simpler than computation with English linear measures?

Lesson 91

Topic: The Metric System of Measurement

Aim: To understand the metric system of weight and of liquid measurement

Specific Objectives:

Metric units of weight measure
Metric units of liquid measure
Relationship among metric units of length, weight, and liquid capacity

Materials needed: A set of metric weights; ounce weights; a balance scale; a liter container

Motivation: The label of a bottle of vitamin pills, which contained minerals as well, showed the iron content of each pill to be 10 milligrams.

What is the meaning of a milligram?

I. Procedure

A. Metric units of weight measure

1. Review metric linear units.

a. What is the basic unit of linear measure in the metric system?

b. Have pupils recall the meaning of the prefixes used in the metric system.

c. Show the similarity between a table that shows place value in our numeration system and a table that shows the relationship among metric linear units.

thousands	hundreds	tens	ones	tenths	hundredths	thousandths

kilometer	hectometer	dekameter	meter	decimeter	centimeter	millimeter

2. Introduce the gram as the basic unit of weight in the metric system.

   a. Discuss with pupils the fact that just as we use a standard unit length to measure length, so also do we use a standard unit weight to measure weight.

   b. What are some standard units of weight measure in our English system of measurement? (pound, ounce, ton)

   c. Have pupils recall the following equivalents:

      16 ounces = 1 pound
      2000 pounds = 1 ton

   d. Tell pupils that in the metric system the basic unit of weight is the gram. Show pupils several gram weights (1 gram; 2 grams; 5 grams, and so on). Compare these weights with the ounce weights.

      Note: The metric unit for measuring weight is defined as the weight of water (at a certain temperature) contained in the volume of a cube whose edge is 1 centimeter long. Thus, the weight of 1 cubic centimeter of water is called a gram.

   e. Have pupils estimate in grams the weights of various objects, and have them check these estimates by using the balance scale.

3. Using their knowledge of metric prefixes, have pupils suggest the meaning of the terms: decigram (dg); centigram (cg); milligram (mg); kilogram (kg).

4. Help pupils develop a table of metric weight equivalents such as the following:

1 milligram (mg)	=	.001	gram
1 centigram (cg)	=	.01	gram
1 decigram (dg)	=	.1	gram
1 gram	=	1	gram
1 decagram (dkg)	=	10	grams
1 hectogram (hg)	=	100	grams
1 kilogram (kg)	=	1000	grams

5. Have pupils see the interrelationships among metric units of weight.

   a. 10 mg = 1 cg or 1 mg = .1 cg
      10 cg = 1 dg or 1 cg = .1 dg, and so on
      10 dg = 1 g
      10 g = 1 dkg
      10 dkg = 1 hg
      10 hg = 1 kg

   b. Replace the frames.

      1 mg = □ cg = □ dg = □ g
      1 cg = □ mg = □ dg = □ g
      1 dg = □ mg = □ cg = □ g = □ dkg
      1 g = □ mg = □ cg = □ dkg = □ hg = □ kg
      1 kg = □ g = □ dkg = □ hg

6. Elicit that units of weight in the metric system, like the units of metric linear measure, are based on powers of ten. We therefore compute with these measures in a manner similar to computation with whole numbers.

B. Metric units of liquid measure

1. What are some standard units of liquid measure in the English system? (pint, quart, gallon)

2. Elicit the equivalents:

   2 pints = 1 quart
   4 quarts = 1 gallon

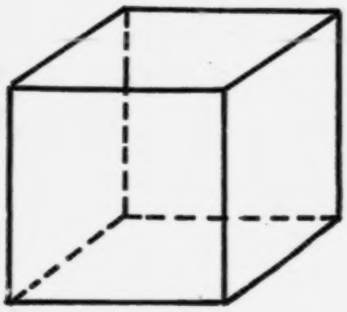
3. Tell pupils that in the metric system the basic unit of liquid measure is the liter. Show pupils a liter container and compare it with a quart container.

4. Elicit the meaning of milliliter (ml); centiliter (cl); deciliter (dl); kiloliter (kl), and develop a table of equivalents as in A-5.

C. Relationship among the metric units of length, weight, and liquid measure

1. Consider a cube whose edge is 1 centimeter long. The weight of water (at 4° C) contained in the volume of such a cube is 1 gram.

2. Consider a cube whose edge is 10 centimeters long.



a. The capacity of this cube (one with edge 10 centimeters in length) is defined to be a liter.

b. Tell pupils that such a cube can be filled with a thousand 1-centimeter cubes. Therefore, the weight of the water needed to fill such a cube is 1000 grams.

c. How many milliliters does a 10-centimeter cube contain?

d. What is the weight of 1 milliliter of water at 4° C? (1 gram)

II. Practice

A. Express the following in grams.

1. 10 cg
2. 150 cg
3. 25 mg
4. 5 dg
5. 8 kg

B. Express the following in liters.

1. 5 ml
2. 35 ml
3. 20 cl
4. 7 dl
5. 2 kl

C. Jars used in chemical laboratories are graduated in milliliters. How many liters does a jar marked 1500 milliliters hold?

D. Perform the indicated operation.

1. 6 grams 3 decigrams
+4 grams 9 decigrams

2. 7 kilograms 3 grams
-5 kilograms 5 grams

3. 2 liters 8 deciliters
x 5

4. 3)4 kiloliters 50 liters

III. Summary

A. What is the basic unit of weight in the metric system? the basic unit of liquid measure?

B. Name other units of weight in the metric system. Explain how each is related to the basic unit.

C. Name other units of liquid measure in the metric system. Explain how each is related to the basic unit.

D. What is the weight of the quantity of water (at 4° C) that would fill a cube whose edge is 1 centimeter long?

E. What are the dimensions of a cube which holds 1 liter?

F. What new vocabulary did you learn today?

(gram, liter)

Lesson 92

Topic: The Metric System of Measurement

Aim: To compare metric and English units of linear measure

Specific Objectives:

Converting from metric units of measure to English units and vice versa
Solving simple problems involving conversion of units from one system to another

Challenge: Which is the greater distance, and by how much, a 100-yard dash or a 100-meter dash?

I. Procedure

A. Converting from one system to the other

1. Compare metric and English units of linear measure.

a. Have pupils compare the length of the meter stick with that of the yardstick. They discover that the meter is approximately 40 inches (39.37 in.).

1) Compute the approximate English equivalent in yards for 1 meter.

$1 \text{ m} = 39.37 \text{ in.}$(approximately)

$1 \text{ m} = \frac{39.37}{36} \text{ yd.}$

$1 \text{ m} = 1.1 \text{ yd.}$(approximately)

2) Compute the approximate metric equivalent for 1 yard.

$1 \text{ yd.} = 36 \text{ in.}$

$1 \text{ yd.} = \frac{36}{39.37} \text{ m}$

$1 \text{ yd.} = .9 \text{ m}$ (approximately)

3) Refer to challenge question.

$100 \text{ m} = 100 \times 1.1 \text{ yd.}$
$100 \text{ m} = 110 \text{ yd.}$

The 100-meter dash is approximately 10 yards longer than the 100-yard dash.

b. Have pupils use their metric rulers to compare the length of the centimeter with that of the inch. They observe that a centimeter is slightly less than $\frac{1}{2}$ inch.

1) Compute the approximate English equivalent for 1 centimeter.

$1 \text{ m} = 39.37 \text{ in.}$

$1 \text{ cm} = \frac{39.37}{100} \text{ in.}$

$1 \text{ cm} = .4 \text{ in.}$ (approximately)

2) Compute the approximate metric equivalent for 1 inch.

$39.37 \text{ in.} = 1 \text{ m}$

$39.37 \text{ in.} = 100 \text{ cm}$

$1 \text{ in.} = \frac{100}{39.37} \text{ cm}$

$= 2.5 \text{ cm}$ (approximately)

c. Have pupils compare the length of the kilometer with that of the mile.

1) Compute the approximate English equivalent for 1 kilometer.

$1 \text{ m} = 1.1 \text{ yd.}$ or 3.3 ft. (approximately)

$1 \text{ km} = 3300 \text{ ft.}$

$= \frac{3300}{5280} \text{ mi.}$

$= .6 \text{ mi.}$ or $\frac{5}{8}$ mi. (approximately)

2) Find the approximate metric equivalent for 1 mile.

1 mi. = 5280 ft. (approximately)

$= \frac{5280}{3300}$ km

= 1.6 km (approximately)

2. Have pupils summarize the following approximations:

Metric to English	English to Metric
1 km = .6 or $\frac{5}{8}$ mi.	1 mi. = 1.6 km
1 m = 40 in. (39.37 in.)	1 yd. = .9 m
1 cm = .4 in.	1 in. = 2.5 cm

B. Solving problems involving conversion

1. Pose problem: Jack's pen pal in Belgium wrote to Jack that he is 150 centimeters tall. Jack's height is 61 inches. Who is taller, and by how much?

   a. Elicit that:

      1) to solve the problem requires the operation of subtraction;

      2) both measurements must be expressed in the same unit - centimeters or inches - before the comparison can be made.

   b. Express 150 centimeters as an equivalent number of inches.

      1 cm = .4 in.
      150 cm = 150 x .4 in.
      150 cm = 60 in.

   c. Jack is taller by approximately 1 inch.

2. Pose problem: The distance from the earth to the sun is approximately 93,000,000 miles. Express this distance in kilometers.

   a. Estimate: Since a mile is greater than a kilometer, the distance will be greater than 93,000,000 kilometers.

b. 1 mile = 1.6 kilometers
93,000,000 miles = 93,000,000 x 1.6 kilometers
= 148,800,000 kilometers

3. Have pupils practice solving simple problems involving conversion of measures from one system to another.

II. Practice

A. Convert each of the following measurements as indicated.

1. 10 cm = □ in.
2. 50 m = □ yd.
3. 85 km = □ mi.
4. 150 mi. = □ km
5. 15 in. = □ cm
6. 22 yd. = □ m
7. 7.5 cm = □ in.
8. 125 mm = □ in.

B. The speed limit in a South American city is 40 km per hour. What is the speed limit in miles per hour?

C. How many centimeters long is a 12-inch ruler?

D. Which is longer, and by how much, the 440-yard dash or the 400-meter dash?

E. What is the width in inches of a 16 mm film? of a 35 mm film?

III. Summary

A. What is the English equivalent of a meter? a kilometer? a centimeter?

B. What is the metric equivalent of an inch? a yard? a mile?

C. What difficulties would be encountered if the United States decided to change over from the English system of measurement to the metric system?

Lesson 93

Topic: The Metric System of Measurement

Aim: To compare metric and English units of weight measure; of liquid measure

Specific Objectives:

Converting from metric units of weight measure to English units, and vice versa
Converting from metric units of liquid measure to English units, and vice versa

Challenge: A French housewife bought a half kilo of pears.
What is the weight of the pears in our system of measurement?

I. Procedure

A. Converting from one system of weight measure to the other

1. Have pupils collect labels on which the weight of the article is stated in English units and in metric units.

Have them make a listing such as the following:

Item	Weight	
Gelatin	3 oz. or	85 grams
Tomato sauce	8 oz.	227 grams
Ripe olives	$8\frac{1}{2}$ oz.	241 grams
Pears	1 lb. 13 oz.	822 grams
Breakfast oats	2 lb.	907.18 grams

2. Metric equivalents of English units

a. The ounce

3 oz. = 85 gm (first item)
1 oz. = 28 gm (approximately)

Thus, one gram is $\frac{1}{28}$ of an ounce, a very small weight.

b. The pound

8 oz. = 227 gms (second item)
16 oz. or 1 lb. = 454 gms (approximately)

3. English equivalents of metric units

   The kilogram

   a. 1 kg = 1000 gm; 1 lb. = 454 gm

      How many 454's are in 1000?

      1000 ÷ 454 = 2.2

      Thus, 1 kg = 2.2 lb. (approximately)

   b. Tell pupils that in countries using the metric system, the kilogram is the commonly used unit of measure in the daily lives of people, because the gram is too small for ordinary purposes. In these countries, the kilogram is commonly referred to as the kilo.

   c. Refer to challenge problem.
      A half kilo of pears is approximately 1.1 lb. in our system of weight measurement.

B. Converting from one system of liquid measure to the other

1. Have pupils collect labels on which the liquid content of the article is stated in English units and in metric units.

   Have them list items:

Item	Liquid Content	
Cleaning fluid	1 qt.	or .946 liters
Pineapple juice	1 pt.2 oz.(fluid)	.53 liters
Rug cleaner	16 oz.(fluid)	.47 liters
Tomato juice	1 qt.14 oz.(fluid)	1.36 liters

2. Have pupils note from the listing that

   a. a quart is a little less than a liter.

      1 quart = .9 liters (approximately)

      or

   b. a liter is a little more than a quart.

      1 liter = 1.06 qt. (approximately)

II. Practice

A. Convert each of the following measurements as indicated.

1. 15 qt. = □ ℓ
2. 16 pt. (fluid) = □ ℓ
3. 2 kg = □ lb.
4. 2 oz. = □ gm
5. 8.8 lb. = □ kg
6. 5 ℓ = □ qt.

B. Florence weighs 100 lb. When she weighed herself on a metric scale during a recent visit to France, did the scale indicate 50 kg, more than 50 kg, or less than 50 kg? Explain.

C. Compute your weight in the metric system. What unit will you use?

D. Francois, who lives in Paris, drinks a liter of milk a day.
Bill, in New York, drinks a quart of milk a day.
How does the amount of milk that Francois drinks compare with the amount that Bill drinks? Explain.

E. About how many quarts is the liquid measure of a cube whose side is 10 cm?

III. Summary

A. What is the English equivalent of a kilogram?

B. What is the English equivalent of a liter?

C. Why do we say "approximately" whenever we convert from one system of measurement to another?

Lesson 94

Topic: Linear Measure

Aim: To review and reinforce the concept of the perimeter of a polygon

Specific Objectives:

Review of concept of polygon
Review of meaning of perimeter of a polygon
Finding the perimeter of some equilateral polygons

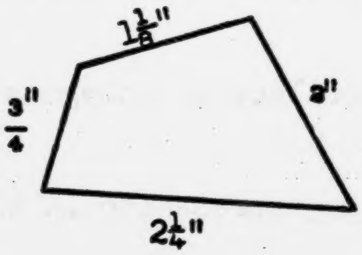
Motivation: Lou's father is planning to buy fencing for the rectangular garden plot in back of the house. What does he need to know about the plot to decide how much fencing he should buy?

I. Procedure

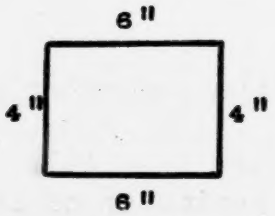
A. Review of concept of polygons

1. Which of the following drawings represent polygons?

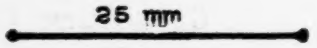
a.



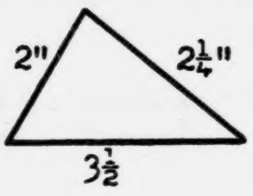
b.



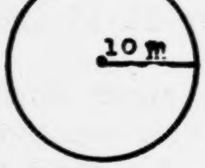
c.



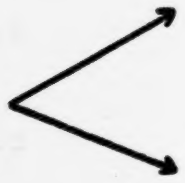
d.



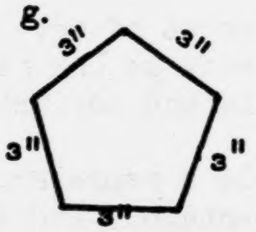
e.



f.



g.



2. Why did you decide that the figures in a, b, d, and g represent polygons?

3. Classify the polygons according to the number of their sides. (quadrilateral, triangle, pentagon)

4. What special kind of quadrilateral is represented in figure b? (rectangle)

5. What special kind of polygon is represented in figure g? (equilateral pentagon)

6. What kind of polygon is suggested by a "stop" sign, a baseball diamond, a snowflake?

B. Review of meaning of perimeter of a polygon

1. Refer to figures in A-1.
What is the sum of the measures of the sides of the polygon represented in figure a? in figure b? in figure d? in figure g?

2. How do we refer to the sum of the measures of the sides of a polygon? (perimeter)

3. What is the rule for finding the perimeter of a polygon? (We add the measures of the sides to find the perimeter of a polygon.)

C. Review finding the perimeter of some equilateral polygons

1. Refer to figure g in A-1.
What is the perimeter of the pentagon represented in this figure?

   a. Add the measures of the five sides. $3+3+3+3+3=15$
   The perimeter is 15 inches.

   b. In what way, other than adding the measures of the sides, can we find the perimeter of this pentagon? (Multiply the measure of a side by 5.)

2. If the measurement of one side of an equilateral pentagon is 6 inches, what is the measurement of each of the other sides? What is the perimeter of the equilateral pentagon?

3. If the variable s represents the measure of the side of an equilateral pentagon, and the variable p represents the measure of the perimeter, what is the meaning of the sentence $p = 5 \times s$ or $p = 5s$? Is this sentence true for all equilateral pentagons?

4. Have pupils recall that a sentence such as $p = 5s$ is called a formula.

5. Help pupils develop a formula for the perimeter of each of the following polygons:

an equilateral triangle	($p = 3 \times s$ or $p = 3s$)
a square	($p = 4s$)
an equilateral hexagon	($p = 6s$)
an equilateral octagon	($p = 8s$)

6. Review the formula for the perimeter of the rectangle.

   a. Elicit the methods the pupils used to find the perimeter of the polygon (rectangle) represented in figure b of A-1.

      1) Add the measures of the four sides. $4+6+4+6 = 20$
         The perimeter is 20 inches.

      2) Add twice the measure of two adjacent sides.
         $(2\times4) + (2\times6) = 8+12 = 20$
         The perimeter is 20 inches.

      3) Twice the sum of the measures of two adjacent sides.
         $2\times(4+6) = 2\times10 = 20$
         The perimeter is 20 inches.

   b. Express the rules as formulas using the variables p, ℓ, and w where

      p represents the measure of the perimeter
      ℓ represents the measure of one side
      w represents the measure of the adjacent side

      1) $p = \ell+w+\ell+w$

      2) $p = (2\times\ell) + (2\times w)$

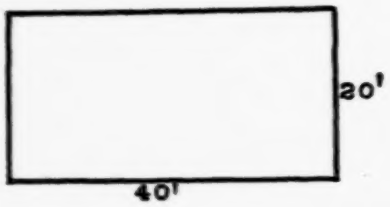
      3) $p = 2\times(\ell+w)$

   c. Answer the motivation problem.

7. Have pupils see how formulas may be used to help solve problems involving perimeter.

   a. Pose problem: How much fencing is needed to enclose a rectangular garden 40' × 20'? (read: 40' by 20')

b. Draw diagram:



c. What formula can you use for finding the perimeter?

d. Solve, using the formula:

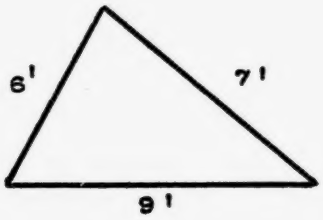
$$\begin{aligned} p &= (2\times \ell) + (2\times w) \\ &= (2\times 40) + (2\times 20) \\ &= 80 + 40 \\ &= 120 \end{aligned}$$

The amount of fencing needed is 120'.

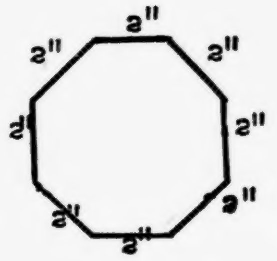
II. Practice

A. Find the perimeter of the polygon represented in each of the following figures.

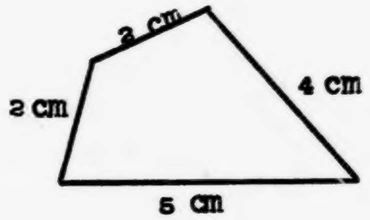
1.



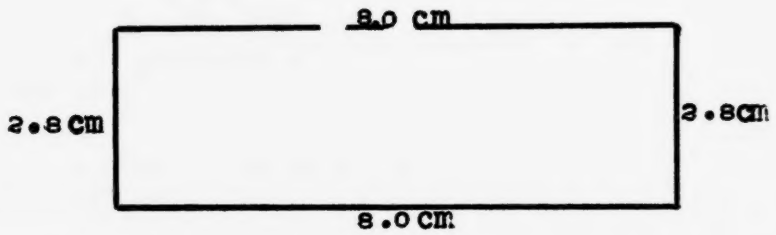
2.

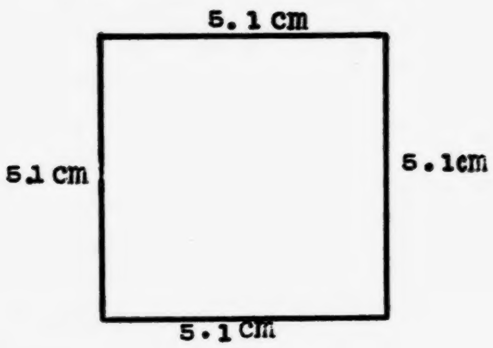


3.



B. Which of the polygons pictured below has the greater perimeter?





C. A baseball diamond is in the shape of a square, of which the side is 90'. What distance is covered by a player who makes a home run?

D. Bill plans to make a rectangular picture frame which will be $15\frac{3}{4}$" long and $8\frac{1}{2}$" wide. How many feet of frame should he buy?

E. Find the cost of placing a fence around a rectangular plot of ground 100' by 35', at 80¢ a foot.

F. A square measures 25' on a side. Make a drawing of the square using the scale 1" = 10'. (1 inch represents 10 feet)

1. What is the perimeter of the given square?

2. What is the perimeter of the square represented in the scale drawing?

3. What is the ratio indicated in the scale? ($\frac{1}{120}$)

4. What is the ratio of the two perimeters?

G. What might be the measures of the sides of a rectangle whose perimeter is 100 cm? (30 and 30; 40 and 10; 33 and 17; 9½ and 40½, and so on)

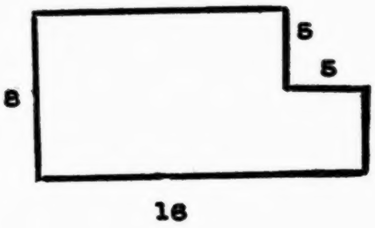
H. What might be the measure of the side of a square whose perimeter is 100 cm? (25 cm - the only possibility)

I. Mr. Keller paid $120 to have a fence placed around his garden, which is 12 feet square. What was the cost per foot of fencing the garden?

J. Helen is planning to edge a tablecloth with lace. The cloth is rectangular in shape and measures 72" by 108". If she allows two additional inches for each corner, how many yards of lace does she need?

K. What is the perimeter of a hexagonal tile whose side is one inch?

L. (Optional) What is the perimeter of the figure in the diagram?



M. (Optional) Develop a formula for the perimeter of a triangle which has two congruent sides.

N. (Optional) What mathematical principle can be used to show that $2l + 2w = 2x(l + w)$?

III. Summary

A. What must you know in order to find the perimeter of a polygon?

B. How do we refer to a polygon with congruent sides?

C. What is the rule for finding the perimeter of any polygon?

D. What is the rule for finding the perimeter of an equilateral polygon?

E. What is the formula for the perimeter of a square? of a rectangle?

F. What is the relationship expressed in the formula $p = 4s$? in the formula $p = 2\ell + 2w$?

G. What mathematical vocabulary did you review today?

(polygon, perimeter, equilateral)

Lessons 95 and 96

Topic: The Circle

Aim: To review and extend understanding of some terms and concepts relating to the circle

Specific Objectives:

- Meaning of circle
- Line segments and circles
- Central Angle
- Arc

Motivation: Why does a wheel run smoothly?

I. Procedure

A. Review of meaning of circle

1. What is a circle?

2. What is the difference between a circle and a polygon?

3. Why is the circle a special kind of plane closed curve?

4. Which space figures that you know have a circle as a face? (cylinder, cone)

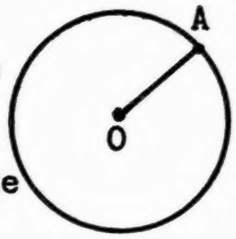
5. What instrument is used for drawing a circle?

6. What do we call the fixed point that is equally distant from each point on the circle?
   Where is this point located? (inside the circle)

B. Review of line segments and circles

1. The radius

a. Using compasses, have pupils draw a circle with point O as center. Have them select any point A on the circle, and draw line segment OA. What name do we give to a line segment such as this? (radius)



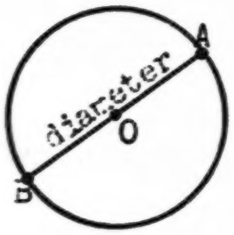
b. Where are the endpoints of any radius located?

c. Draw several radii of circle O.
How many radii are there for a given circle? (an unlimited number)

d. How do the measures of the radii of a given circle compare?
(They are the same.)

e. Have pupils discuss the motivation question.

2. The diameter



a. Have pupils extend $\overline{AO}$ to intersect the circle at B.

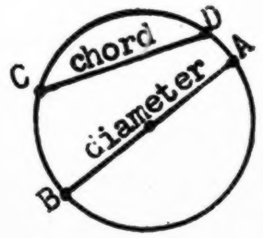
What name do we give to a line segment such as $\overline{AB}$? (diameter)

b. Where are the endpoints of any diameter?
What other point does a diameter contain?

c. How many diameters are there for a given circle?

d. How do the measures of the diameters of a given circle compare?

3. The chord



a. Have pupils choose two other points, C and D, on the circle. Have them draw $\overline{CD}$. How do we refer to a line segment such as this? (chord)

b. Where are the endpoints of any chord?

c. How many chords can be drawn in a circle?

d. Is $\overline{AB}$ a chord? Explain.

e. Draw several additional chords in circle O.
Which is the longest chord of a circle?

f. Relationship between the radius and diameter of a circle.

1) Have pupils measure the two radii, $\overline{BO}$ and $\overline{OA}$.
Have them measure the diameter, $\overline{BA}$.

2) Draw a picture of any other diameter EF.
How long is $\overline{EO}$? $\overline{OF}$? $\overline{EF}$?

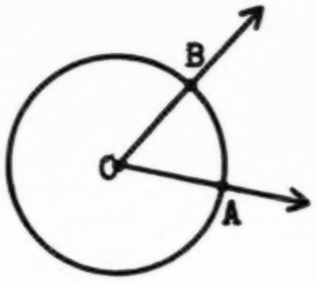
3) Have pupils conclude that a diameter of a circle has twice the measure of any radius of the same circle.

4) Letting d represent the measure of the diameter in unit lengths, and r the measure of the radius in unit lengths, elicit that the relationship of the diameter to the radius can be expressed by the following formula: $d=2r$ or $r=\frac{1}{2}d$.

5) What is the measure of the diameter of a circle whose radius measures $1\frac{1}{2}$ inches?
What is the measure of the radius of a circle if the diameter is 2.6 centimeters in length?

C. Review of meaning of central angle

1. Have pupils consider a drawing such as the one at the right.



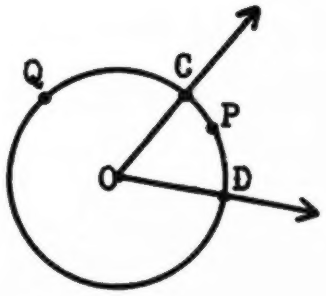
a. What kind of geometric figure is formed by $\overrightarrow{OB}$ and $\overrightarrow{OA}$ meeting at O?

b. Where is the vertex of the angle placed?

c. What name do we give to such an angle? (central angle)

2. Select and name a point which is on the circle and in the interior of the angle; on the circle and in the exterior of the angle.

D. Review of meaning of arc



1. In the picture of the circle at the right, consider the two subsets formed by central angles COD. What do we call each of these subsets? (an arc)

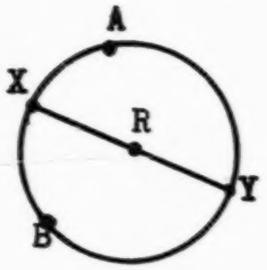
2. How do we refer to the arc which is the union of C and D and the points on the circle in the interior of $\angle COD$? (minor arc)

3. How do we refer to the arc which is the union of C and D and the points on the circle in the exterior of $\angle COD$? (major arc)

4. Is point P in the major or minor arc?
Is point Q in the major or minor arc?

5. What symbol can we use for minor arc CD? ($\overset{\frown}{CD}$ or $\overset{\frown}{CPD}$) for major arc CD? ($\overset{\frown}{CQD}$)

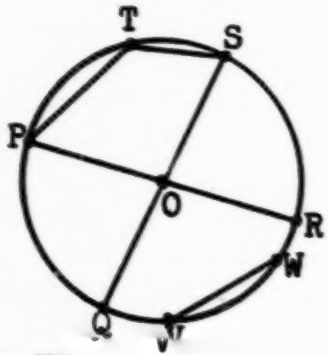
6. What name do we give to an arc which has two points at the end of a diameter and the remaining points lying on one side of the diameter? (semi-circle)



7. In the picture at the right, $\overset{\frown}{XAY}$ is one semi-circle. What is the other?

8. How does the length of a semi-circle compare with the length of the circle? What is the meaning of semi?

II. Practice



A. Refer to the figure at the right, which pictures a circle with center O.

1. Name: all the diameters; all the radii; all the chords; four arcs; two semi-circles; 4 central angles; 4 angles which are not central angles.

2. Which is the longest chord in the circle? ($\overline{PR}$, $\overline{QS}$)

3. The length of $\overline{OR}$ is one inch. What is the length of $\overline{OP}$?

4. What line segments are congruent to $\overline{OR}$?

B. Use "always" or "sometimes" or "never" to fill the blank space, so that a true statement results.

1. The radius of a circle is _____ a chord.

2. The diameter of a circle is _____ a chord.

C. The diameter of a circle is 3" in length. What can you say about every chord in the circle that is not a diameter? (It is less than 3" in length.)

D. What is the measure of the diameter of a circle if the measure of the radius is:

1. 2 in. 2. $3\frac{1}{4}$ ft. 3. 1.5 cm

E. What is the measure of the radius of a circle if the measure of the diameter is:

1. 5 cm 2. $4\frac{1}{2}$ in. 3. 9.6 ft.

F. A chord of a circle is 5 inches in length. What can you tell about the length of the diameter? (The diameter is greater than 5 inches unless the chord being considered is the diameter.)

III. Summary

A. How do you recognize a circle?

B. Where are the endpoints of a radius of a circle located? the endpoints of a diameter? the endpoints of a chord?

C. How do you know that there are an infinite number of radii in a circle? an infinite number of diameters?

D. How do the measures of the radii of a circle compare? the measures of the diameters?

E. What is the relationship between the diameter of a circle and the radius?
Express the relationship in a mathematical sentence.

F. What is the meaning of a central angle of a circle?

G. What is meant by an arc of a circle?

H. What mathematical vocabulary did you review today?

(radius, diameter, chord, central angle, arc, semi-circle)

Lesson 97

Topic: Linear Measurement

Aim: To discover the relationship between the circumference of a circle and its diameter

Specific Objectives:

Meaning of circumference
Need for indirect measurement of circumference
Discovery of an approximate relationship between the circumference of a circle and its diameter

Challenge: What distance is covered in one complete revolution of a bicycle wheel that is 28 inches in diameter?

I. Procedure

A. Meaning of circumference

1. What name is given to the length of a polygon? (perimeter) How could we find the perimeter of an object such as a rectangular tablecloth?

2. Have pupils consider how we would determine the amount of trimming needed for the rim of a circular lampshade. They may suggest wrapping a piece of string around the rim until the ends just meet and then measuring the length of the string with a ruler.

3. Tell pupils that the length of a circle is called the circumference of the circle.

B. Need for indirect measurement of circumference

1. Review the distinction between direct and indirect measurement.

   a. A direct measurement involves directly comparing an object to be measured with a unit of measurement. For example, the length of a textbook may be measured directly with a ruler.

   b. Measurements may also be made indirectly by measuring a related quantity. Thus, we may find the measure of the perimeter of a rectangle indirectly by first performing a direct measurement of its length and its width, and then using the formula $p = 2\ell + 2w$ to compute the perimeter.

2. Elicit that it is difficult to measure the length of a circular object directly. Therefore, there is a need for finding a method of measuring its circumference indirectly.

C. Relationship between the circumference of a circle and its diameter

1. Have pupils draw pictures of circles of diameter 1 inch; 2 inches; 3 inches.

    a. What happens to the size of the circumference as the diameter increases?

    b. Have pupils conclude that there is a relationship between the circumference of a circle and its diameter.

2. Consider the drawings on the right.

Figure 1

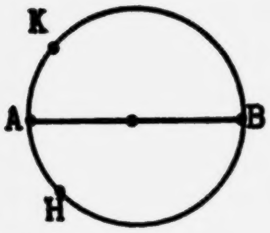
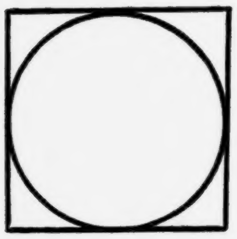


Figure 2



    a. In Figure 1, which appears to be longer, $\overline{AB}$ or $\widehat{AKB}$? $\overline{AB}$ or $\widehat{AHB}$?

        1) Elicit that the circumference of a circle is greater than the combined lengths of two diameters.

        2) Express this conclusion symbolically: $c > 2 \times d$ or $c > 2d$.

    b. Consider Figure 2 of C-2. Which appears to be longer, the distance around the model of the circle, or the distance around the model of the square that encloses the circle?

        1) Elicit that the perimeter of the square is a total of 4 diameters and is greater than the circumference of the circle.

        2) Express symbolically: $c < 4d$.

3. Have pupils "guess" that the length of the circumference of a circle is approximately three times the length of its diameter.

$$c \approx 3d$$

4. On the basis of this "guess," answer the challenge question. The distance covered in one complete revolution of a 28-inch bicycle wheel is approximately 3×28 or 84 inches.

5. Have pupils realize that the hypothesis (guess) about the relationship between the circumference of a circle and its diameter needs confirmation.

6. Homework: Ask pupils to find and record the lengths of the diameters and circumferences of several circular objects all of different sizes, such as a lampshade, a wastebasket, a plate, and so on. A tape measure or piece of string may be used for this purpose. This information will be used in the next lesson.

II. Practice

A. What is the length of the diameter of a circle whose radius measures $1\frac{1}{4}$"? 2.4 cm? $7\frac{1}{8}$'?

B. What is the length of the radius of a circle whose diameter measures 20'? 6.8 yd.? 4.32 m?

C. When we find the diameter of a circle by means of the formula $d = 2r$, are we using direct or indirect measurement? Which is the direct measurement? Which is the indirect measurement?

D. Of two circles, one with a diameter of $1\frac{1}{4}$", the other with a diameter of 4 cm, which has the greater circumference? Explain.

E. What is the approximate length of the circumference of a circle whose diameter measures 1"? whose radius measures 1"?

F. Find the approximate length of the circumference of a circle whose diameter measures 5'; 14 cm; 26 in.

III. Summary

A. What is the equivalent of "perimeter" when speaking of a circle?

B. When you measure the width of a room with a yardstick, are you measuring directly or indirectly? Is finding the distance to the sun an example of direct or indirect measurement?

C. When you use the formula $p = 2l + 2w$ to compute the perimeter of a rectangle, are you measuring directly or indirectly? Which measurements have to be made directly before the formula can be used? (measurements of length and width)

D. What is the approximate relationship between the circumference of a circle and its diameter?

E. How can you make an indirect measurement of the circumference of a circle? Which measurement would have to be made directly?

F. What mathematical vocabulary did you use today?
(circumference, direct measurement, indirect measurement)

Lesson 98

Topic: Linear Measurement

Aim: To introduce the formula $c = \pi \times d$ for the circumference of a circle

Specific Objectives:

The meaning of π and an approximation for π
The formula $c = \pi \times d$ for the circumference of a circle
Use of the formula to solve problems

Motivation: How can we tell more precisely (than we were able to in the previous lesson) the distance covered in one revolution of a wheel that is 28" in diameter?

I. Procedure

A. The meaning and approximate value of π

1. Tabulate the results of the measurements made by pupils in doing the homework assignment of the previous lesson.

Circular object	Circumference	Diameter
Basin	38"	12"
Pail	31"	10"
Wheel	63"	20"
Record	22"	7"

2. Have pupils compute the ratio of each circumference to the length of its diameter.

Circular object	Circumference	Diameter	$c \div d$
Basin	38"	12"	$38 \div 12 = 3.16+$
Pail	31"	10"	$31 \div 10 = 3.10$
Wheel	63"	20"	$63 \div 20 = 3.15$
Record	22"	7"	$22 \div 7 = 3.14+$

3. Elicit that in each case the ratio is approximately equal to 3.1.

$$c \approx 3.1 \times d$$

4. Tell pupils that the ratio of the circumference of any circle to the length of its diameter is always the same. This ratio, $\frac{c}{d}$, has been computed to several thousand decimal places and it has never terminated or repeated. For this reason it has been given a special name, the Greek letter pi or π (pronounced pie).

5. Tell pupils that approximate values of π that are commonly used are 3.14 and $3\frac{1}{7}$. Elicit that any computation performed by using an approximation for π will lead to an approximate result.

B. The formula $c = \pi \times d$ for the circumference of a circle

1. Have pupils recall that in a previous lesson they found that the circumference of a circle is approximately equal to three times the length of its diameter.

2. How could we now express the relationship of the circumference of a circle to the length of its diameter? (The circumference of a circle is π times the length of the diameter.)

$$c = \pi \times d \text{ or } c = \pi d$$

3. Have pupils realize that when we use the formula $c = \pi \times d$ to compute the circumference of a circle, we are performing indirect measurement. Which direct measurement is needed for the indirect measurement?

4. If we know the length of the radius of a circle, how can we find the circumference?

   a. Elicit that since $d = 2r$, we may double the radius to find the diameter and then use the formula $c = \pi \times d$.

   or

   b. Since $d = 2r$, we can replace d by 2r in $c = \pi \times d$, obtaining $c = \pi \times 2r$. Using the associative and commutative properties, we obtain $c = 2\pi r$, the usual form when r is used.

C. Use of the formula

1. Consider the motivation problem: The diameter of a bicycle wheel is 28". How far does the wheel travel in one revolution?

   a. Why does one revolution of the wheel represent the circumference of a circle?

   b. Solution: $c = \pi \times d$

$$\begin{aligned} c &\approx 3\tfrac{1}{7} \times 28 \\ &\approx \frac{22}{7} \times 28 \\ &\approx 88 \end{aligned}$$

   The distance covered by the wheel in one revolution is approximately 88".

c. Solve the problem using 3.14 as an approximation for π.

$$c = \pi \times d$$
$$c \approx 3.14 \times 28$$
$$\approx 87.92$$

The distance covered by the wheel is approximately 87.92".

d. Have pupils compare results in b and c.

1) Elicit that the difference in the distances, when both approximations for π are used, is .08 of an inch. Since the difference is slight, we may use whichever approximation is more convenient.

2) Which approximation for π was easier to work with? Why? Have pupils conclude that when the measure of the diameter is a multiple of 7, the value $3\frac{1}{7}$ is more convenient for computation.

e. Answer the motivation question.

2. Pose problem: The length of the earth's radius is approximately 4000 miles. What is the approximate length of the equator?

a. Elicit that we wish to find the circumference of the circle represented by the earth's equator. The length of the radius of this circle is approximately 4000 miles.

b. Solution:

$$\begin{array}{lcl} c = 2\pi \times r & \text{or} & c = \pi \times d \\ \approx 2 \times 3.14 \times 4000 & & \approx 3.14 \times 8000 \\ \approx 25{,}120 \approx 25{,}000 & & \approx 25{,}120 \approx 25{,}000 \end{array}$$

The length of the equator is approximately 25,000 miles.

II. Practice

A. Approximate the circumference. Use $\pi \approx 3.14$ or $3\frac{1}{7}$, whichever is more convenient. The length of the diameter is

1. 14 in. b. 10 cm c. 44 ft. d. 4.9 mm

B. Approximate the circumference. Use $\pi \approx 3.14$ or $3\frac{1}{7}$. The length of the radius is

1. 50 yd. b. 10.5 m c. 250 mi. d. 2.1 ft.

C. Jack plans to put a wire screen around a circular flower bed which is 10 feet in diameter. How many feet of screen does he need?

D. Scientists are planning to assemble an orbiting space station in the form of a wheel 350 feet in diameter. What will be the circumference of the proposed wheel?

E. The distance from the foot of a maypole to the circle formed by the children dancing around it is 10 feet. Ruth skipped around the pole once. What distance did she skip?

F. (Optional) If the circumference of a circle is 132 inches, what is the length of the diameter? of the radius?

III. Summary

A. What ratio does π express?

B. What are the approximate values of π commonly used?

C. What formula expresses the relationship between the circumference of a circle and the length of its diameter?

D. If you know the length of the radius of a circle, how can you find the circumference?

Lessons 99 and 100

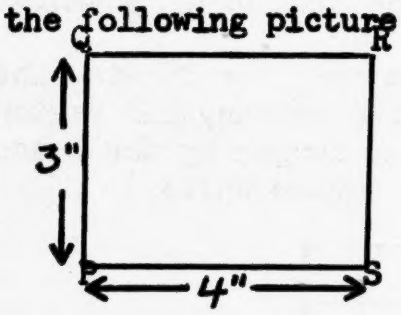
Topic: Square Measure

Aim: To review and extend concepts of area measurement

Specific Objectives:

The difference between linear and square measure
Review of formula for the area of a rectangular region
Formula for the area of a square region
Solving problems involving area

Challenge: Consider the following picture of a rectangle:



The number 12 and the number 14 describe two measures of rectangle PQRS. What measures?

I. Procedure

A. Review of difference between linear and square measure

1. Name the four line segments that are the sides of rectangle PQRS.

2. What kind of unit is used to measure a line segment? Why?

3. How do we compute the perimeter of a rectangle?

4. What is the perimeter of rectangle PQRS?

5. Into how many sets of points does the rectangle divide the plane? (three - the set of points outside the figure, or the exterior; the set of points which form the figure; the set of points inside the figure, or the interior)

6. Recall that the union of a rectangle and its interior is called a rectangular region.

7. What do we call the measure of the interior of a region? (area)

8. What kind of unit is used to find the area of a plane region? (We must use a unit plane region. By common agreement, we use a square unit region.)

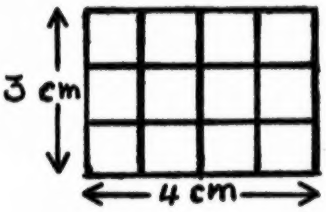
9. What are some standard square unit regions? (square inch, square foot, square mile, square centimeter, and so on)

10. What is the area, in square inches, of rectangular region PQRS?

11. Answer the challenge question.

B. Review of formula for the area of a rectangular region

1. Have pupils state the rule for finding the area of a rectangular region. (The area of a rectangular region is found by multiplying the measure of the length by the measure of the width. The area is expressed in square units.)



$3 \times 4 = 12$

Area is 12 square centimeters

2. What is the formula for finding the area of a rectangular region? ($A = \ell \times w$) What does each symbol in the formula represent?

3. Find the area of a rectangular playground which is 212 feet long and 205 feet wide. Solve by rule or formula.

   a. $A = \ell \times w$
      $= 212 \times 205$
      $= 43{,}460$

      The area is 43,460 square feet.

   b. Are you measuring directly or indirectly when you use the formula? What direct measurements are made before the formula can be used?

C. Formula for the area of a square region

1. What name do we give to a rectangle where each side has the same measure?

2. Have pupils recall that in finding the perimeter of a square region, the measure of a side may be represented by s. Elicit the formula

$A = s \times s$ for the area of a square region, instead of $A = \ell \times w$.

a. If the measure, in inches, of the side of a square is 10, what is the area?

$$A = 10 \times 10$$

The area is 100 square inches.

b. Recall that 10×10 may be expressed as 10^2. How can we express $s \times s$? (s^2)

c. Elicit the formula $A = s^2$ for the area of a square region.

3. What is the area of a square region whose side measures 26 centimeters?

$$A = s^2$$
$$A = 26 \times 26 = 676$$

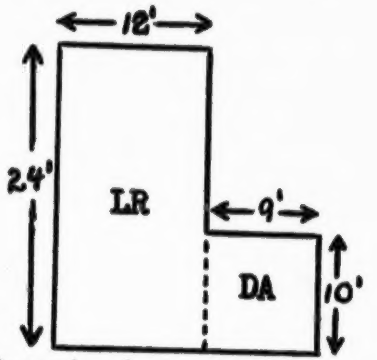
The area is 676 square centimeters.

4. Table of square measure

144 sq. in. = 1 sq. ft.
1296 sq. in. = 1 sq. yd.
9 sq. ft. = 1 sq. yd.

D. Solving problems involving area

1. Problem: A living room with a dining alcove has the shape of two rectangular regions as indicated.



Mrs. Jonas wishes to buy wall-to-wall carpeting for the entire area. At $9 per square yard, what will be the cost of the carpeting?

a. Elicit the need for finding the measure of the region. Then multiply that number by the cost per square yard.

b. Compute: $A = \ell \times w$.

Area of Living Room $= \frac{24}{3} \times \frac{12}{3} = 32$

Area of Dining Alcove $= \frac{9}{3} \times \frac{10}{3} = 10$

Total Area $= 42$ sq. yd.

$42 \times 9 = 378$

c. The cost of the carpeting is \$378.

2. Problem: A baseball diamond is in the shape of a square with side 90 feet long. What is the area of this region?

a. $A = s^2$
$A = 90^2 = 90 \times 90 = 8100$

b. The area of the region is 8100 square feet.

II. Practice

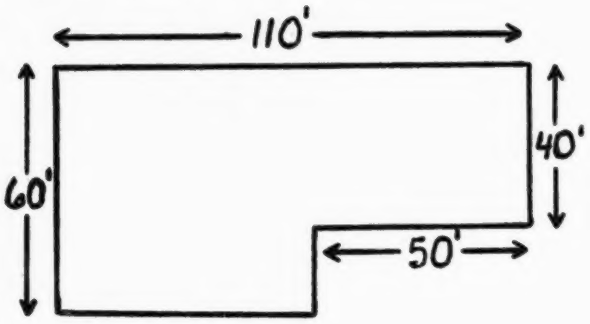
A. Find the area of the rectangular regions which have the following lengths and widths.

	length ℓ	width w
1.	16 ft.	9 ft.
2.	40 yd.	25 yd.
3.	165 cm	80 cm
4.	$7\frac{1}{2}$ in.	6 in.
5.	15.5 cm	12.8 cm

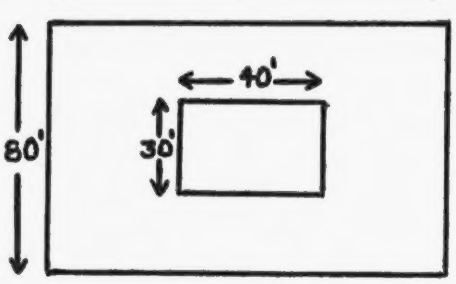
B. Find the area of the regions bounded by squares with sides:

1. 17 in.
2. $8\frac{1}{2}$ yd.
3. 6.5 m
4. 83 mi.
5. 250 ft.

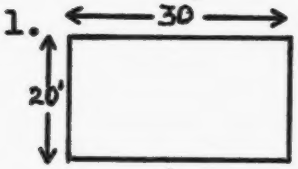
C. A plot of ground in the shape of two rectangular regions is represented by the drawing at the right. Find its area.

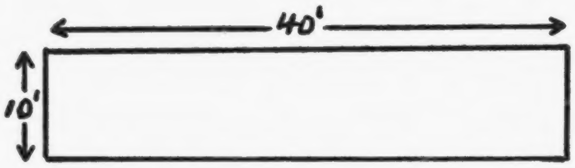


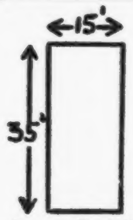
D. A house in the country is built on a rectangular plot of ground which is 120 feet long and 80 feet wide. The house, also rectangular in shape, is 40 feet long and 30 feet wide. What is the area of the ground that is left for lawn and trees?

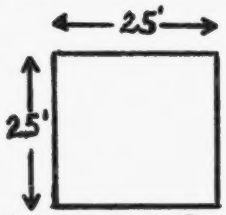


E. Compute the area of the rectangular regions represented below. In each case, the perimeter of the region is 100 feet.

1.


2.


3.


4.


Which of these regions has the greatest area?
What special name do we give to this region? (square region)

Note to teacher: Have pupils experiment with other given perimeters, as for example 60 and 80, to see what they can discover. One conclusion they may come to is that for a given perimeter, the special rectangle we call the square has the greatest area.

III. Summary

A. What is meant by a rectangular region?

B. When is a linear unit used in measurement? a square unit?

C. What are some standard square units of measurement?

D. How do you find the area of a rectangular region?

E. What is the formula for the area of a square region?

F. When you use a formula, are you measuring directly or indirectly, or both? Explain.

G. Why can you use the formula for a rectangular region in finding the area of a square region?

Lesson 101

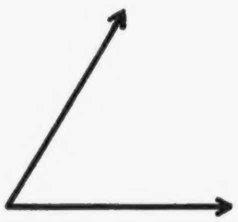
Topic: Angle Measurement

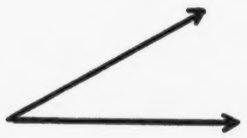
Aim: To develop some basic ideas of angle measurement

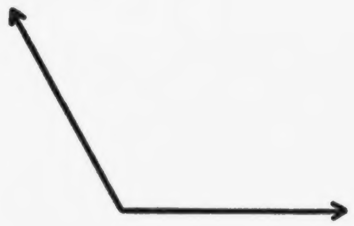
Specific Objectives:

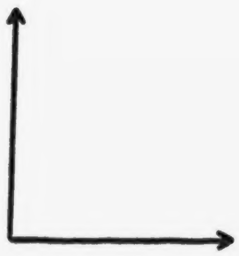
Review of meaning of angle
Need for a unit of angle measurement
Use of a (non-standard) unit angle to measure angles

Challenge: Which of the following angles has the greatest measure?









I. Procedure

A. Review of meaning of angle

1. What is an angle? (the union of two rays with a common end point)

2. Name the angle pictured at the right.

3. What is the vertex of the angle?

4. Name the sides.

5. Have pupils draw an angle, label it, and name it.

B. Need for a unit of angle measure

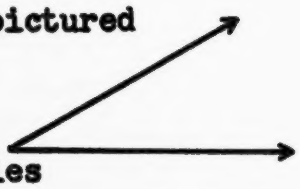
1. Have pupils recall that an angle partitions the plane into three sets of points:

   a. the set of points inside the figure (the interior)
   b. the set of points outside the figure (the exterior)
   c. the set of points on the figure

2. Have them also recall that to measure line segments, we use a unit length; to measure area we use a unit area. Have pupils realize that these units cannot be used to measure any of the sets of points into which an angle divides the plane.

3. Elicit the need for a unit of the same kind as the thing to be measured. Thus, to measure an angle we must use a unit angle.

4. Have pupils see that an angle such as the one pictured at the right could be used as a unit of angle measurement.



C. Use of a (non-standard) unit angle to measure angles

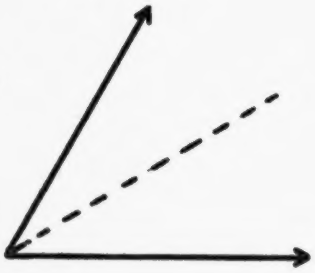
1. Distribute rexographed sheets picturing the four angles of the challenge question.

   (Suggestion: $\angle w$ could be made approximately 60°; $\angle x$, 30°; $\angle y$, 120°; $\angle 2$, 90°. Do not use the word "degree" or the symbol for degrees here, or for any unit angle used in this lesson. The word and the symbol will be introduced in the next lesson.)

2. Distribute a cutout of a unit angle (of 30°) to each pupil.

3. Have pupils measure $\angle w$ by applying the unit angle to $\angle w$ as many times as it will "fit."

   They will see that the unit angle is contained twice in $\angle w$.



4. Tell pupils that we say the measure of $\angle w$, in unit angles, is 2. $m \angle w = 2$.

5. Have pupils measure $\angle x$, $\angle y$, $\angle 2$ in the same way. ($m \angle x = 1$; $m \angle y = 4$; $m \angle 2 = 3$)

6. Answer the challenge question.

7. Distribute another unit angle (of 15°) and have pupils measure the four angles using this unit angle. ($m \angle w = 4$; $m \angle x = 2$; $m \angle y = 8$; $m \angle 2 = 6$)

   Why are the results of measurement with the second unit angle used different from those obtained when the first unit angle was used?

8. Have pupils conclude that a standard unit of angle measurement is needed.

II. Practice

Practice similar to C-3 to 8 is suggested. (The rexographed sheet distributed to the pupils could have many more than the four angles studied for pupils to measure.)

III. Summary

A. What kind of unit of measure is used in measuring angles?

B. How is a unit angle used to measure angles?

C. If a unit angle is used to measure angle ABC, what would "$m \angle ABC = 5$" mean?

D. Why is it necessary to have a standard unit angle?

E. What new vocabulary did you learn today?

(unit angle)

Lessons 102 and 103

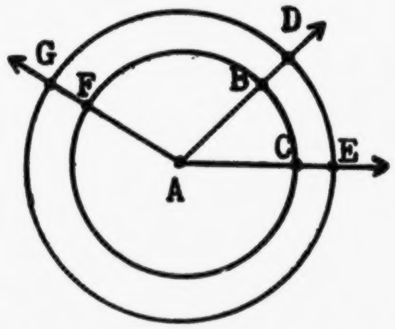
Topic: Angle Measurement

Aim: To learn to measure angles with a protractor; to draw pictures of angles of specified measures

Specific Objectives:

Visualizing any angle as a central angle of a circle
Meaning of degree
Understanding the markings on a protractor
Use of a protractor

Challenge: Consider the following drawing which represents two circles with the same center, A.



Which central angle of the outside circle is the same as central angle BAC of the inside circle? as central angle FAB of the inside circle?

I. Procedure

A. Visualizing any angle as a central angle of a circle

1. Elicit the answer to the challenge question.

2. Does minor arc BC of the inside circle appear to have the same measure as minor arc DE of the outer circle? (no)

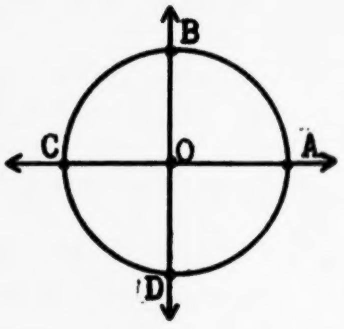
3. Suppose the measure of minor arc BC is one-sixth of the circumference of the inner circle. What do you think is true of the measure of minor arc DE in relation to the circumference of the outer circle?

4. How many pictures of circles can we draw with A as center?

5. Have pupils see that any angle can be thought of as a central

angle of many circles. For each such circle, the ratio of the measure of the minor arc determined by the angle to the circumference of the circle is the same.

B. Meaning of degree



1. In the figure at the right, points A, B, C, and D have been chosen so that minor arcs AB, BC, CD, and DA are all equal in measure.

   What appears to be true of the central angles determined by these arcs? (They have the same measure, that is to say, they are congruent.)

2. Suppose we partitioned a circle into 360 minor arcs all of the same measure. What would be true of the central angles determined by these arcs?

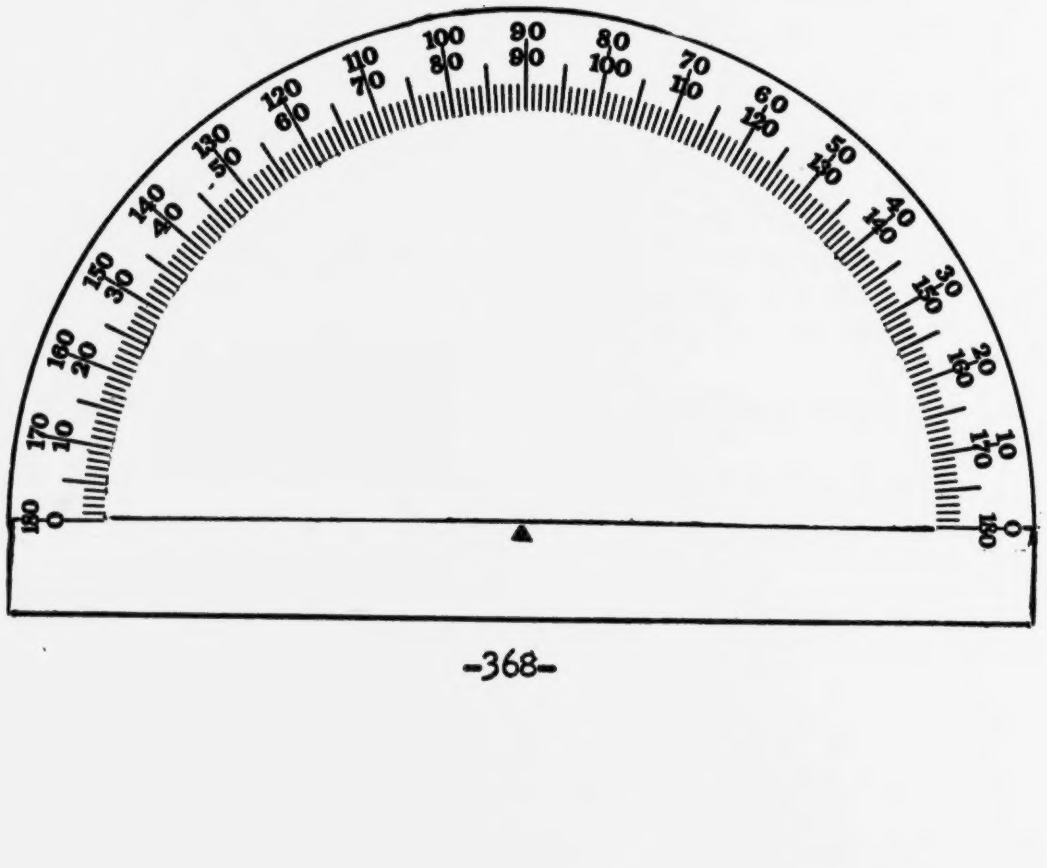
3. Tell pupils that each of these (360) central angles measures 1 degree. We usually write this as 1°. The degree is the standard unit of angle measurement.

   We often say there are 360° in a circle.

C. Understanding the markings on the protractor

1. Tell pupils that an instrument called a protractor is used for measuring angles represented on paper.

2. Have pupils study their protractors. Discuss the instrument. (The overhead projector can be very helpful here.)



a. A protractor is a representation of a segment together with a semicircle having the segment as a diameter. The semicircle is partitioned into 180 minor arcs which are equal in measure.

b. These minor arcs determine 180 unit angles of one degree each. The common end-point of the rays (the sides of the angles) is represented on the protractor, usually by an arrowhead.

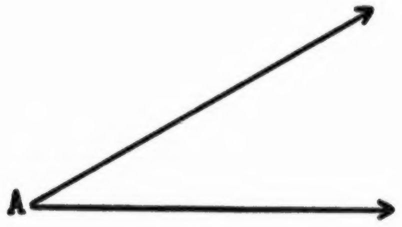
c. Although we consider the rays to be numbered in order from 0 to 180 inclusive, only the first ray represented, and every tenth one thereafter, have numerals placed alongside.

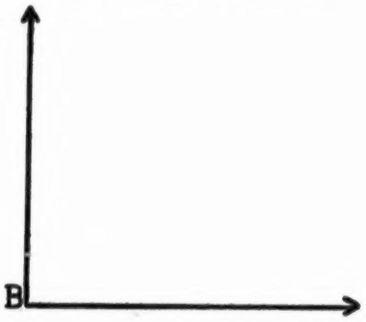
d. The angle indicated by any two successive markings on the protractor is an angle of 1 degree.

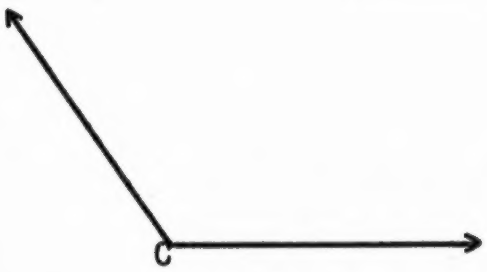
e. For convenience in measuring, most protractors have two scales, one reading from 0 to 180 in one case from right to left, and on the other scale from left to right.

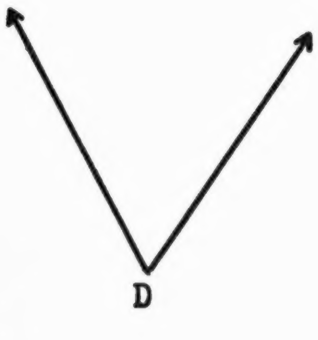
D. Use of the protractor

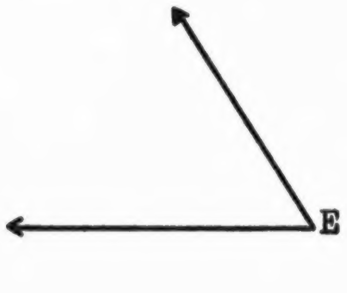
1. To help pupils learn to use a protractor to measure angles represented on paper, it is suggested that a rexographed sheet be distributed containing drawings of several angles, such as the following:





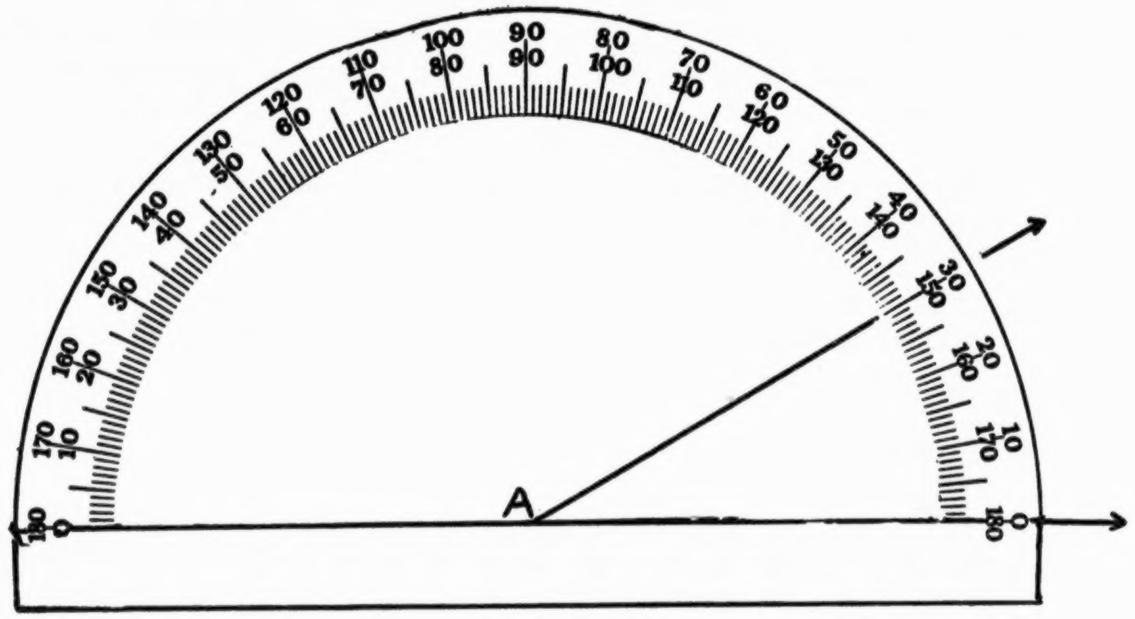






a. To measure $\angle A$, we proceed as follows:

1) Place the protractor so that the arrowhead is at the vertex of the angle and one side of the angle corresponds to the ray marked 0. See the figure below.



2) Read the scale whose zero point lies on one side of the angle to find to which ray the other side of the angle corresponds. (the ray marked 30)

Note: If the sides of the angle in a drawing are not long enough to reach the scale, extend them.

3) The number 30 is the number of degrees in the angle. The number itself is called the measure of $\angle A$. To indicate that $\angle A$ has a measure of 30, we write $m\angle A = 30$.

4) We can also express the fact that the measure of $\angle A$ is 30 by writing:

$\angle A = 30^\circ$ or

$\angle A$ is a 30° angle

Note: If a circular protractor were used to measure $\angle A$, its measure could be read as 30 or $360-30=330$. To avoid ambiguity, we agree at this grade level to use the lesser number as the measure. Thus, the measure of every angle will be equal to or less than 180.

b. Have pupils follow the procedures in a-1 to 4 as they find the measures of angles B, C, D, and E.

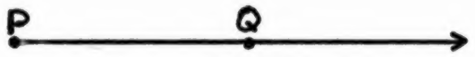
c. Have pupils practice using their protractors to measure a variety of angles to the nearest degree.

d. Elicit the greatest possible error of measurement in using such protractors.

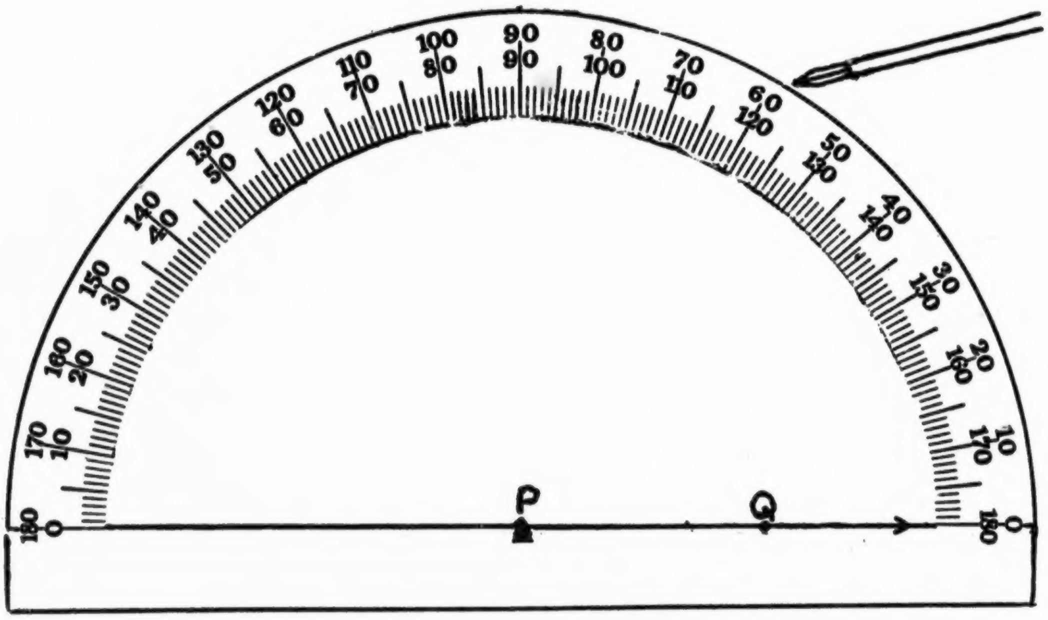
2. Have pupils learn to use the protractor to draw pictures of angles of specified measures.

   a. Draw a picture of an angle with a measurement of 60°.

      1) Since an angle is formed by two rays with a common end point, we start with a ray, such as $\overrightarrow{PQ}$.



      2) Then place the protractor along the ray so that the center of the protractor (arrowhead) is at the end point, P, of the ray, and the zero mark on the protractor is on the ray. See the figure below.



3) Mark point R at the correct 60° mark on the protractor.

4) Draw $\overrightarrow{PR}$.

5) ∠QPR is a 60° angle.

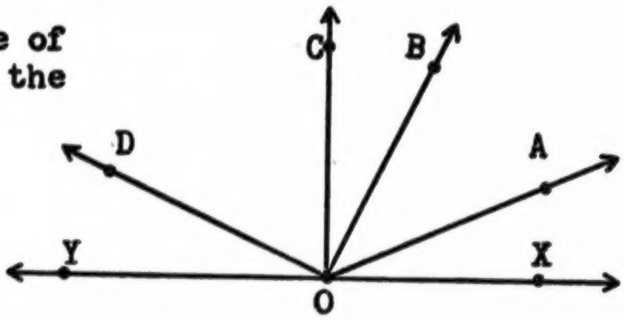
b. Have pupils use their protractors to draw pictures of angles of various specified measures.

II. Practice

A. What is the measure of a central angle determined by an arc that is $\frac{1}{3}$ of a circle? $\frac{1}{4}$ of a circle? $\frac{1}{5}$ of a circle? $\frac{1}{12}$ of a circle?

B. Draw pictures of four angles of different size. Use your protractor to find their measures to the nearest degree.

C. Use a protractor to tell the measure of the following angles represented in the diagram at the right



∠XOA ∠XOB ∠AOB ∠XOC

∠BOD ∠XOY

D. Refer to the diagram in C and replace the frames.

1. $m\angle AOC + m\angle COD = \square$
2. $m\angle XOC + m\angle COY = \square$
3. $m\angle XOY - m\angle XOB = \square$

E. Use a protractor to draw pictures of angles of:

1. 50° 2. 125° 3. 69° 4. 182°

III. Summary

A. How many circles can be thought of as having a given angle as a central angle?

B. What is the standard unit of angle measurement?

C. What is the ratio of the minor arc determined by a 1° central angle to the circumference of the circle?

D. What is meant by the measure of an angle?

E. What instrument do we use to measure angles?

F. Describe the steps in drawing an angle of a specified measure.

G. What new vocabulary did you learn today?

(degree, protractor)

Lesson 104

Topic: Angle Measurement

Aim: To learn that angles may be classified according to their size

Specific Objectives:

Meaning of right angle; acute angle; obtuse angle; straight angle
Meaning of perpendicular lines

Challenge: Consider the figures below.

Figure 1

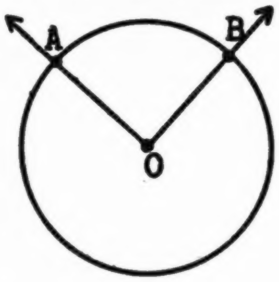
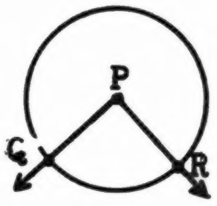


Figure 2



In Figure 1, points A and B are so located that $\widehat{AB}$ is $\frac{1}{4}$ of the circle. In Figure 2, $\widehat{QR}$ is $\frac{1}{4}$ of the circle. How does the measure of $\angle AOB$ compare with the measure of $\angle QPR$?

I. Procedure

A. Classification of angles as right, acute, obtuse, and straight

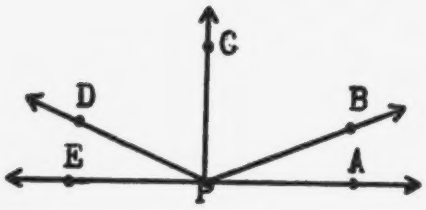
1. Meaning of right angle

a. Elicit that although the sides of $\angle QPR$ in Figure 2 of the challenge appear to be "shorter" than the sides of $\angle AOB$ in Figure 1, the measures of the two angles are the same. Since the minor arc determined by each angle in its circle is $\frac{1}{4}$ of the circle, each is a 90° angle.

b. Tell pupils that an angle which has a measure in degrees of 90 is called a right angle.

c. Have pupils give examples of objects in the classroom which suggest right angles (corner of desk; corner of book).

2. Meaning of acute angle; obtuse angle



   a. Have pupils consider a diagram such as the one at the right.

      1) Which angles represented in the diagram seem to have a measure of less than 90? ($\angle APB$, $\angle BPC$, and so on)

      2) Check with a protractor.

   b. Tell pupils that an angle whose measure in degrees is less than 90 is called an acute angle.

   c. Follow similar procedures to develop the meaning of an obtuse angle as an angle whose measure in degrees is between 90 and 180.

3. Meaning of straight angle

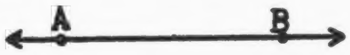
   a. Refer to the diagram in 2-a.

      1) Which angles appear to be right angles? ($\angle APC$, $\angle CPE$) Check with a protractor.

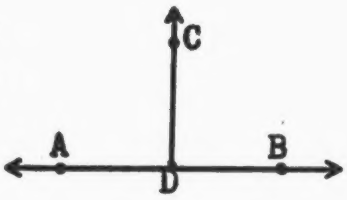
      2) Which angle appears to have a measure that is twice as large as a right angle? ($\angle APE$) What is the measure of $\angle APE$?

   b. Tell pupils that an angle whose measure in degrees is 180 is called a straight angle. Ask pupils why they think this name was selected.

B. Perpendicular lines



1. Have pupils draw a line such as $\overleftrightarrow{AB}$.
   Have them use the corner of a square card to draw a ray, $\overrightarrow{CD}$, from a point D on $\overleftrightarrow{AB}$ so that $\angle ADC = \angle CDB$.



   a. Tell pupils that when a ray meets a line so that two equal angles are formed, we say the ray is perpendicular to the line. The symbol for "is perpendicular to" is $\perp$. $\overrightarrow{CD} \perp \overleftrightarrow{AB}$

   b. What is the measure in degrees of $\angle ADB$? (180)
      If $\angle ADC = \angle CDB$, what is the measure of each? (90)
      Have pupils see that a ray is perpendicular to a line if the ray and the line meet so that at least one of the angles formed is a right angle.

2. Elicit that:

   a. two rays are perpendicular if they form a right angle

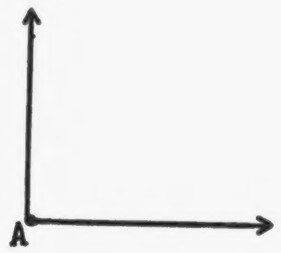
   b. two intersecting lines are perpendicular if any one of the angles formed is a right angle

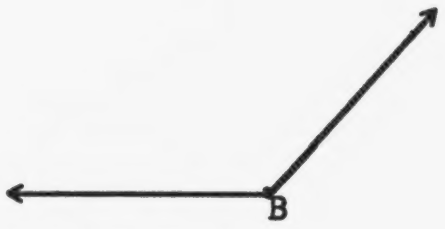
   c. two segments are perpendicular if the lines of which they are a part are perpendicular.

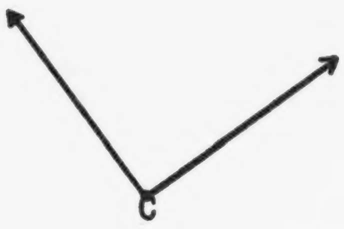
3. Have pupils draw pictures illustrating perpendicular lines, perpendicular rays, perpendicular segments, and combinations of lines, rays, and segments.

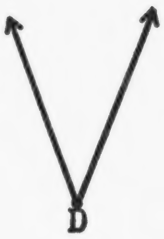
II. Practice

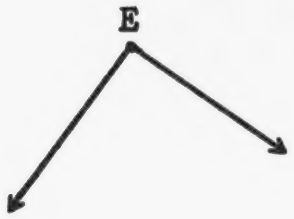
A. Use the square corner of a piece of paper or a card to tell which of the following drawings represent acute angles, right angles, and obtuse angles.

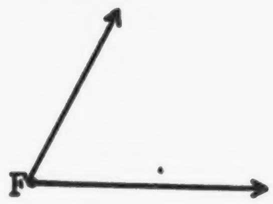






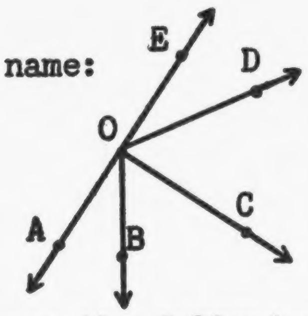






B. In the figure at the right name:

1. 4 acute angles
2. 2 obtuse angles
3. 2 right angles
4. 1 straight angle



C. Classify the angles which have the following measurements.

1. 75°
2. 118°
3. 16°
4. 180°
5. 90°
6. 92°
7. 89°
8. 1°
9. 179°

D. Think of three objects in your home which suggest

1. right angles
2. acute angles
3. obtuse angles
4. straight angles

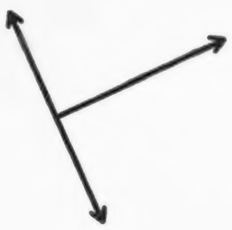
E. What kind of angle is suggested by the hands of a clock when it shows:

1. 3 o'clock
2. 8 o'clock
3. 6 o'clock
4. 12 o'clock
5. 10 minutes after 3
6. 4 o'clock
7. 10 minutes to 11
8. 9 o'clock
9. 25 minutes after 12

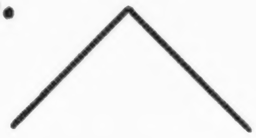
F. Which of the following figures look most like representations of:

1. a ray perpendicular to a segment
2. perpendicular lines
3. a ray perpendicular to a line
4. perpendicular rays
5. perpendicular segments

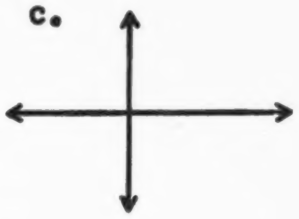
a.



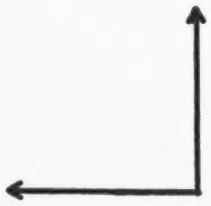
b.



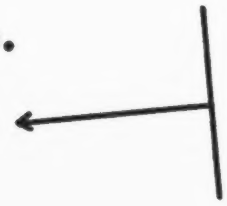
c.



d.



e.



G. What kind of angle is suggested by two rays with a common endpoint, one of which represents a north direction, and the other an east direction? a west direction and a northwest direction? an east direction and a west direction?

H. Draw a picture of two non-perpendicular lines. How many acute angles are formed? How many obtuse angles? How many straight angles?

I. Think of three objects in your home which suggest perpendicular segments.

J. Consider the following pictures of quadrilaterals.

Figure 1

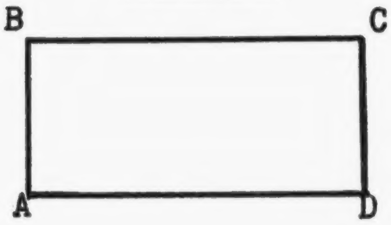
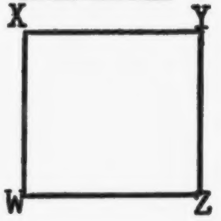


Figure 2



1. In Figure 1, ABCD represents a rectangle.

   a. What is true of the angles of a rectangle?
   b. Name the line segments which are perpendicular to each other. (BA ⊥ AD, AD ⊥ CD, DC ⊥ CB, CB ⊥ BA)

2. In Figure 2, does WXYZ appear to be a rectangle? Explain.

   a. How is WXYZ different from the usual rectangle?
   b. Name the four pairs of line segments which are perpendicular to each other.

III. Summary

A. How do we classify angles? (according to their measure)

B. In terms of measure, how would you describe a right angle? an acute angle? an obtuse angle? a straight angle?

C. When are two lines said to be perpendicular?

D. What new vocabulary did you learn today?

(right angle, acute angle, obtuse angle, straight angle)

What new symbol? (⊥)

Lessons 105 and 106

Topic: Angle Measurement

Aim: To understand some special angle relationships

Specific Objectives:

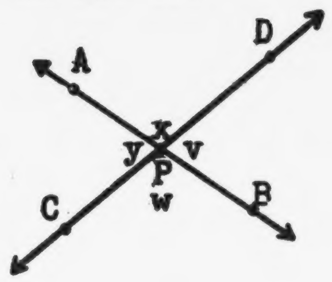
Meaning of adjacent angles
Meaning of vertical angles
Meaning of supplementary angles
Congruency of vertical angles

Challenge: How many angles, other than straight angles, are formed whenever two lines intersect?

I. Procedure

A. Meaning of adjacent angles

1. Have pupils draw a pair of intersecting lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Have them label the point of intersection P.



   a. Refer to the challenge. Elicit that whenever two lines intersect, four angles, other than straight angles, are formed.

   b. Have pupils name the angles formed.

      ∠APD or ∠x

      ∠DPB or ∠v

      ∠BPC or ∠w

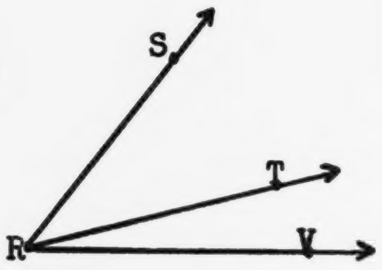
      ∠CPA or ∠y

   c. Have them note that ∠v and ∠x have a common vertex, P, and a common ray PD, but no interior points in common. Tell pupils that such angles are called adjacent angles.

2. Name three other pairs of adjacent angles in the figure (∠v and ∠w; ∠w and ∠y; ∠y and ∠x). For each pair, have them name the common vertex and the common ray.

3. Why are angles SRT and TRV pictured at the right considered to be adjacent angles?

   Are angles SRT and SRV adjacent angles? Explain.



B. Meaning of vertical angles

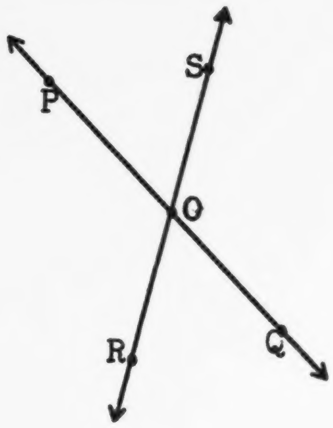
   1. Refer to the diagram in A-1.

      a. Name two non-adjacent angles formed by the intersecting lines. (x and w; y and v)

      b. Tell pupils that any two non-adjacent angles formed by two intersecting lines are called vertical angles. We sometimes describe these angles as being "opposite" each other.

   2. Have pupils draw several pairs of intersecting lines. Have them select and name the vertical angles formed.

C. Meaning of supplementary angles



   1. Consider the diagram at the right.

      a. Name a pair of adjacent angles ($\angle POS$ and $\angle SOQ$) Use your protractor to find the sum of their measures. (180)

      b. Name another pair of adjacent angles? ($\angle POR$ and $\angle ROQ$) Use your protractor to find the sum of their measures. (180)

      c. Name a third pair of adjacent angles ($\angle SOQ$ and $\angle QOR$). Without measuring, what do you think is the sum of their measures? (180)

      d. Tell pupils that two angles are called supplementary angles when the sum of their degree measures is 180. Each angle is said to be the supplement of the other.

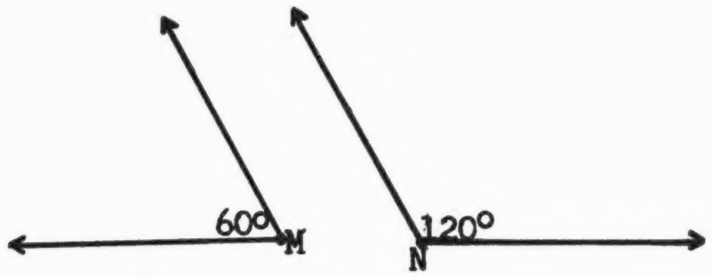
         $\angle POS$ and $\angle SOQ$ are supplementary angles

         $\angle POR$ and $\angle ROQ$ are supplementary angles

         $\angle SOQ$ and $\angle QOR$ are supplementary angles

2. Have pupils see that when adjacent angles each have a ray on the same line but pointing in opposite directions, the angles are supplementary.

3. Tell pupils that two angles can be supplementary without being adjacent. Any two angles whose measures total 180 are supplementary.

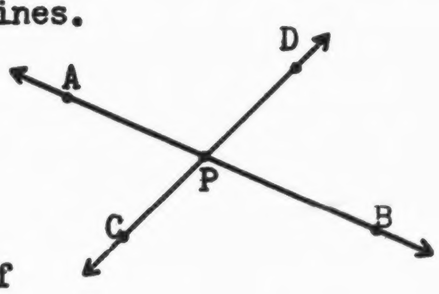


$$m\angle M + m\angle N = 60 + 120, \text{ or } 180$$

Therefore, $\angle M$ and $\angle N$ are supplementary angles.

D. Vertical angles are congruent

1. Have pupils draw a pair of intersecting lines.



   a. How many pairs of vertical angles are formed? Name them.

   b. Measure each angle.

   c. What do you notice about these pairs of vertical angles?

2. Have pupils draw several other pairs of vertical angles and compare their measures.

3. Elicit that vertical angles appear to be congruent.

4. (OPTIONAL) Have pupils see why vertical angles must be a pair of congruent angles.

   a. Refer to the diagram in 1 above. How are angles APC and APD related? (they are supplementary angles)

      1) Suppose $\angle APC$ measures 30°. What is the measure of $\angle APD$? (180-30, or 150)

      2) Suppose $\angle APC$ measures 40°. What is the measure of $\angle APD$? (180-40, or 140)

      3) Let us represent the measure of $\angle APC$ by n. What is the measure of $\angle APD$? (180-n)

b. How are angles APC and CPB related? (they are supplementary angles)

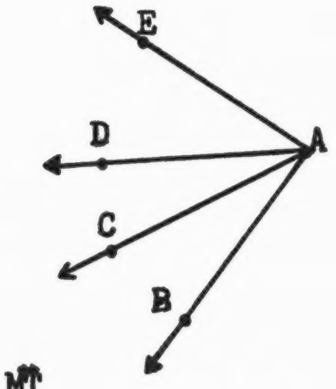
If the measure of ∠APC is represented by n, what is the measure of ∠CPB? (180-n)

c. How are angles APD and CPB related? (they are vertical angles)

The measure of ∠APD is 180-n and the measure of ∠CPB is 180-n. What must be true of these (or any) vertical angles? (they are congruent for they have the same measure)

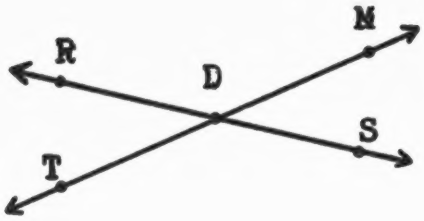
II. Practice

A. Consider the diagram at the right.



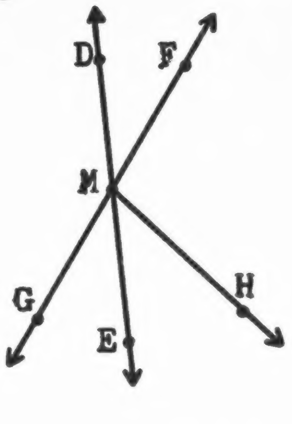
1. Name two pairs of adjacent angles.
2. Are angles EAD and CAB adjacent angles. Explain.
3. Are angles BAC and BAE adjacent angles? Explain.

B. Refer to the figure at the right in which $\overleftrightarrow{RS}$ and $\overleftrightarrow{MT}$ intersect at D.



1. Name two angles that are adjacent to ∠TDS.
2. Name two angles that are adjacent to ∠MDS.
3. Name two pairs of vertical angles.
4. If ∠MDS measures 25°, what is the measure of ∠RDT?

C. In the figure at the right, $\overleftrightarrow{DE}$ and $\overleftrightarrow{FG}$ intersect at M.



1. Are angles DMF and EMH vertical angles? Explain.
2. Name a pair of vertical angles. (∠DMF and ∠GME)
3. Suppose the measure of ∠GME is 40. What is the measure of ∠DMF?

4. Name a pair of supplementary angles. (DMF and GMD)

5. Can you say that $\angle EMH$, $\angle HMF$, and $\angle FMD$ are supplementary? Explain.

D. The measurements of four angles are given below. What is the measure of the supplement of each?

1. 50° 2. 125° 3. 90° 4. 179°

E. If two angles are supplementary and have the same measure, what is the measure of each?

III. Summary

A. Under what condtions are two angles adjacent?

B. What is meant by vertical angles?
What appears to be true of a pair of vertical angles?

C. When is a pair of angles considered to be supplementary?
Must supplementary angles be adjacent?

D. What new vocabulary have you learned today?

(adjacent angles, vertical angles, supplementary angles)

CHAPTER X

In this chapter procedures are suggested for extending the pupil's understanding of per cent as a ratio; for reinforcing his understanding of the close relationship which exists among decimal numerals, fractions, and per cent forms; and for providing practice in using per cents in socially significant problems.

Per cent was originated in connection with business practices in the later Middle Ages. It developed as a tool of commerce and industry, rather than as a tool of mathematics. From a mathematical viewpoint, however, the relationship or ratio idea of per cent is the most meaningful approach to the concept of per cent. But in its practical everyday usage, the interpretation of per cent as a fractional number (expressed as a fraction or as a decimal) is more frequent. Indeed, for computational purposes, the latter approach is often more convenient. For this reason, the materials in this chapter emphasize the close relationship which exists among decimal numerals, fractions, and per cent forms.

Per cent may be thought of in terms of one hundred. Any per cent can be thought of as a ratio with a second term of 100. If we are given a ratio whose second term is not 100, this ratio can be expressed as an equivalent ratio with a second term of 100 in per cent form. For example, if we wish to express the ratio $\frac{30}{50}$ in per cent form, we find that $\frac{30}{50} = \frac{60}{100}$. Thus, $\frac{30}{50}$ is equivalent to 60%. Ultimately, when the meaning of proportion is developed, as well as the basic algebraic techniques for solving open sentences, the use of the ratio approach will make possible a unified handling of all cases of per cent. The usual "three cases of per cent" will not require separate analysis.

In addition to understanding the meaning of per cent as a ratio, pupils should gain an understanding of and skill in the interpretation of per cent as a number. The idea of a per cent as a fractional number is related to its standard usage in business and in commerce. It is also likely that pupils will have encountered this interpretation in their daily lives - when making purchases of articles on sale, and so on. These materials, therefore, suggest procedures for a meaningful development of the two aspects of per cent: per cent as a ratio (a pair of numbers); and per cent as a number. It is hoped that pupils will be guided to realize that there is more than one possible approach to per cent.

It is advisable to avoid rules which involve "moving the decimal point" as pupils benefit most from applying the techniques of expressing per cent ideas consciously and thoughtfully. Most pupils develop their own "short cuts" when they are ready.

The percentage idea has a useful and well-recognized application to certain fields and to certain kinds of problems. Those developed in these materials are:

1. problems in discount
2. problems in commission
3. problems in profit and loss
4. problems in interest

Chapter X

RATIONAL NUMBERS: PER CENT FORM

Lessons 107-116

Lessons 107 and 108

Topic: Per cent

Aim: To develop increased understanding of per cent as a ratio

Specific Objectives:

Review of meaning of per cent as a ratio of a number to the number 100
Expressing ratios in fractional form as per cents
Expressing ratios in decimal form as per cents

Challenge: The school store reported that 20 shirts of its stock of athletic shirts were sold. What per cent of the shirts did the store sell?

I. Procedure

A. Meaning of per cent as a ratio

1. Elicit that since we do not know the number of shirts originally stocked, we cannot answer the challenge question.

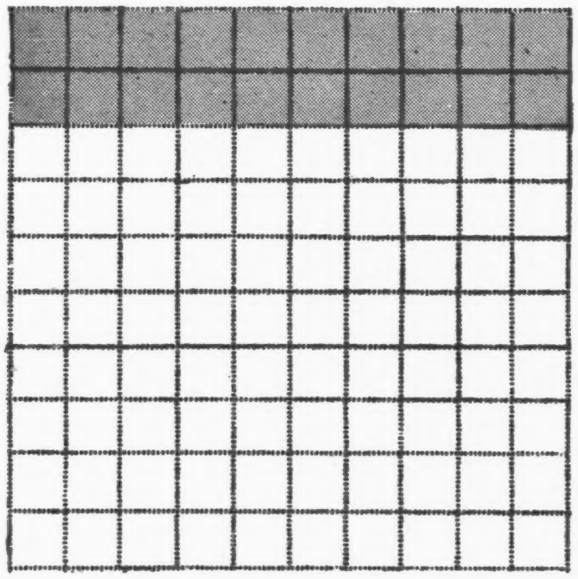
2. If the number of shirts originally stocked is 100, what is the ratio of the number sold to the number stocked? (20 to 100, or 20:100, or $\frac{20}{100}$)

3. Elicit that another name for the ratio $\frac{20}{100}$ is 20 per cent, or 20%.

   Per cent means the ratio of a number to 100. It is based upon the Latin words, "per centum," meaning "per hundred."

4. Have pupils recall that any ratio which compares a number to the number 100 may be called a per cent.

Note to Teacher: If necessary, the above relationship may be visualized by a diagram and a development such as the following:



All 100 squares represent the set of 100 shirts originally stocked. The 20 shaded squares represent the set of 20 shirts sold.

What is the number of the set of shirts sold? What is the number of the set of shirts originally stocked?

What is the ratio of the number of shirts sold to the number of shirts stocked?

(20 to 100, or 20: 100, or $\frac{20}{100}$)

Tell pupils that the ratio of a number to the number 100 may be expressed in another form called per cent. The symbol "%" means "per cent." Thus, the ratio of 20 to 100 may be expressed as 20%.

5. Have pupils express the following ratios in per cent form.

   a. 1 to 100 (1 per cent, or 1%)
   b. 3 to 100
   c. 11 to 100
   d. 50 to 100
   e. $4\frac{1}{2}$ to 100 ($4\frac{1}{2}$%)
   f. $\frac{1}{2}$ to 100 ($\frac{1}{2}$%)
   g. 105 to 100 (105%)
   h. .6 to 100 (.6%)

6. Have pupils express the following per cents as the ratio of a number to 100.

   a. 10% (10 to 100)
   b. 18%
   c. 75%
   d. 62.5% (62.5 to 100)
   e. $33\frac{1}{3}$% ($33\frac{1}{3}$ to 100)
   f. 150%
   g. .5%

B. Expressing ratios in fractional form as per cents

1. Refer to the challenge problem. If the number of shirts originally stocked is 50, what per cent of the shirts did the store sell?

   a. What is the ratio of the number of shirts sold to the number of shirts stocked? $(\frac{20}{50})$

   b. Elicit that if we wish to express the ratio $\frac{20}{50}$ in per cent form, we must rename $\frac{20}{50}$ as a fraction with a denominator of 100.

      What is the solution of $\frac{20}{50} = \frac{\square}{100}$?

   c. Review use of the multiplicative identity to rename a fraction.

      $$\frac{20}{50} \times \frac{2}{2} = \frac{40}{100}$$

      Why did we choose $\frac{2}{2}$ as a name for 1?

   d. $\frac{20}{50} = \frac{40}{100} = \square\%$

   e. Returning to the problem in B-1, 40% of the shirts were sold.

2. Rename each of the following in per cent form.

   a. $\frac{1}{2}$ ($\frac{1}{2} \times \frac{50}{50} = \frac{50}{100}$ or 50%)
   b. $\frac{3}{4}$
   c. $\frac{7}{10}$
   d. $\frac{3}{5}$
   e. $\frac{11}{20}$
   f. $\frac{4}{25}$
   g. $\frac{18}{50}$
   h. $\frac{19}{28}$
   i. $\frac{24}{25}$
   j. $\frac{20}{10}$ ($\frac{20}{10} \times \frac{10}{10} = \frac{200}{100}$ or 200%)

3. Rename each of the following in fractional form with the denominator 100. Then simplify the fractional numerals.

   a. 20% ($20\% = \frac{20}{100} = \frac{1}{5}$)
   b. 50%
   c. 25%
   d. 40%
   e. 75%
   f. 2%
   g. 90%
   h. 85%
   i. 200% ($200\% = \frac{200}{100} = 2$)
   j. $\frac{1}{2}\%$ ($\frac{1}{2}\% = \frac{\frac{1}{2}}{100} = \frac{1}{2} \times \frac{1}{100} = \frac{1}{200}$)
   k. $\frac{3}{4}\%$

C. Expressing ratios in decimal form as per cents

1. Consider the ratio of 17 to 100.

   a. What is the per cent form of this ratio? (17%); the fractional form? ($\frac{17}{100}$); the decimal form? (.17)

   b. Elicit that 17%, $\frac{17}{100}$, and .17 all express the ratio 17 to 100.

2. Consider the ratio 115 to 100.

   What is the per cent form of this ratio? (115%)

   What is the fractional form? ($\frac{115}{100}$)

   What is the decimal form? (1.15)

3. Rename each of the following in fractional form, in decimal form, and as a per cent.

Ratio	Fractional Form	Decimal Form	Per cent Form
3 to 100	$\frac{3}{100}$	.03	3%
27 to 100			
1 to 100			
100 to 100	$\frac{100}{100}$	1.00	100%
150 to 100	$\frac{150}{100}$	1.50	150%
$\frac{1}{2}$ to 100	$\frac{\frac{1}{2}}{100}$ or $\frac{1}{200}$	.005	$\frac{1}{2}$%

4. Elicit that decimal fractions expressed in hundredths can immediately be renamed in per cent form. For example,

$.03$ (or $\frac{13}{100}$) = 13%

$.27 = \square\%$

$.38 = \square\%$

$.01 = \square\%$, and so on.

5. How can we find the per cent form of numbers expressed as decimal fractions with denominators other than 100?

   a. How can we rename .135 as a per cent?

      1) Elicit that since a per cent expresses the ratio of a number to the number 100, we must rename .135 as a fraction with denominator of 100.

      2) Guide pupils to see that we may use the multiplicative identity to accomplish this.

      Thus, $.135 = .135 \times 1$

      $.135 = .135 \times \frac{100}{100}$

      $.135 = \frac{13.5}{100}$ or 13.5%

   b. How can we rename .005 as a per cent?

   $.005 = .005 \times 1$

   $.005 = .005 \times \frac{100}{100}$

   $.005 = \frac{.5}{100}$ or .5%

II. Practice

A. Replace the frames.

	Fractional Form	Decimal Form	Per cent Form
1.	$\frac{18}{100}$	□	□
2.	□	.06	□
3.	$\frac{23}{50}$	□	□
4.	□	$.14\frac{1}{2}$	□
5.	$\frac{105}{100}$	□	□
6.	□	□	$26\frac{1}{2}\%$
7.	$\frac{17}{10}$	□	□
8.	□	□	300%
9.	$\frac{15.5}{100}$	□	□
10.	□	□	3.4%
11.	□	□	$\frac{1}{5}\%$

B. What is the ratio of each of the following numbers compared to 100?

6 15 30 $12\frac{1}{2}$ .8

Express each of these ratios in per cent form.

C. Consider the following ordered pairs of numbers:

5, 50; 3, 10; 7, 20; 9, 25; 17, 10

1. What is the ratio of the first number in each pair to the second?
2. Express each ratio as a fraction with a denominator of 100.
3. Express each ratio in per cent form.

D. Rename each of the following in per cent form.

1. $\frac{3}{5}$
2. $\frac{3}{4}$
3. $\frac{1}{2}$
4. $\frac{26}{25}$
5. $\frac{11}{10}$
6. $\frac{12\frac{1}{2}}{100}$
7. $\frac{66\frac{2}{3}}{100}$
8. $\frac{\frac{1}{2}}{100}$

E. On a mathematics test, John solved $\frac{4}{5}$ of the examples correctly. What per cent did he solve correctly?

III. Summary

A. What is another name for the ratio of a number to the number 100?

B. In what forms can a per cent be expressed?

C. How do you rename a ratio expressed as a fraction with a denominator other than 100, as a per cent?

D. How can you tell when a number in fractional form would be expressed as a per cent greater than 100%?

Lesson 109

Topic: Per cent

Aim: To learn how to find a per cent of a number

Specific Objectives:

Finding a per cent of a number
Solving problems involving finding a per cent of a number

Challenge: A school raised $200 through a cake sale. It was decided to donate 30% of it to a charity. How much money was donated to the charity?

I. Procedure

A. Finding a per cent of a number

1. Refer to challenge problem.

a. In what other forms can we express 30%? ($\frac{30}{100}$, .30)

b. What operation is used to find $\frac{30}{100}$ of 200? (multiplication)

What is the solution of $\frac{30}{100} \times 200 = \square$? (60)

The amount contributed to the charity was $60.

c. Elicit that if 30% is renamed in decimal form as .30, we find 30% of 200 as follows:

$$.30 \times 200 = 60 \text{ or } \begin{array}{r} 200 \\ \underline{\times .30} \\ 60.00 \end{array}$$

2. After several such examples, elicit that we may use any one of the equivalent forms for a number in a computation without affecting the result. We use that name for a number which makes the computation easier.

Note to Teacher: If necessary, the preceding procedure may be visualized by diagrams and a development such as the following:

Figure 1

$100

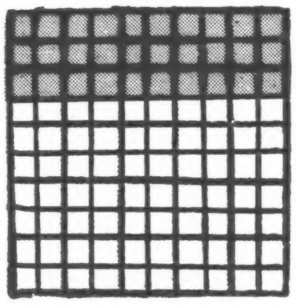
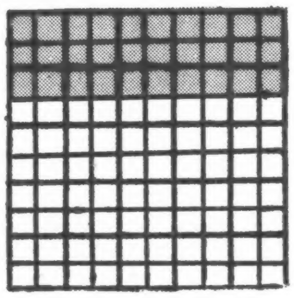


Figure 2

$100



Each square in Figures 1 and 2 represents one dollar. All the squares in Figures 1 and 2 represent the set of two hundred dollars raised.

Shade 30% of Figure 1. How many squares did you shade?
Shade 30% of Figure 2. How many squares did you shade?

How many shaded squares represent 30% of $200? What is 30% of $100? What is 30% of $200?

B. Solving problems

1. Consider problem: A farmer who raises turkeys sold 20% of his flock of 4000 turkeys. How many turkeys did he sell?

   a. Elicit that to determine how many turkeys were sold, we must find 20% of 4000.

      What is the solution of: $\frac{1}{5} \times 4000 = \square$ or $.20 \times 4000 = \square$? (800)

   b. Thus, 800 turkeys were sold.

2. Pose problem: In a school of 1900 pupils, 87% bought tickets for the school spring festival. How many pupils bought tickets?

   a. Elicit that to find the number of pupils who bought tickets, we must find 87% of 1900. In what other forms may we express 87%? What operation is involved?

   b. We compute as follows: $\frac{87}{100} \times 1900 = \square$ or $.87 \times 1900 = \square$.

$$\begin{array}{ll} \frac{87}{100} \times 1900 = \frac{165300}{100} & \text{or} \quad \begin{array}{r} 1900 \\ \underline{\times\ .87} \\ 13300 \\ \underline{15200} \\ 1653.00 \end{array} \\ \phantom{\frac{87}{100} \times 1900} = 1653 & \end{array}$$

   c. Thus, 1653 pupils bought tickets.

3. Dan's family has an income of \$8000 a year. The family plans to try to save $6\frac{1}{2}\%$ of its income during the coming year. How much money is the family planning to save?

   a. To find the amount of money the family plans to save, we will find $6\frac{1}{2}\%$ of \$8000.

      $\frac{13}{200} \times 8000 = \square$ or $.065 \times 8000 = \square$

      $\frac{13}{200} \times 8000 = 520$ or $.065 \times 8000 = 520$

      Which form of the per cent do you consider easier to use?

   b. The family is going to try to save \$520.

II. Practice

A. Compute the following:

1. 40% of 80
2. 12% of 320
3. 9% of 1500
4. $\frac{1}{2}\%$ of 1800
5. 120% of 75
6. $1\frac{1}{2}\%$ of 100

B. The population of a school is 1800. On a certain day 96% were present. How many children were present on that day?

C. In science we learn that approximately 70% of the body weight is water. If Carl weighs 125 lbs., how much of his weight is water?

D. Jack earned \$75. If he plans to save 20% of this, how much does he plan to save?

III. Summary

A. How can we rename a per cent in order to find the per cent of a number?

B. What operation do we use to find a per cent of a number?

Lesson 110

Topic: Per cent

Aim: To use the procedure for finding a per cent of a number in problems involving discount

Specific Objectives:

Meaning of discount; finding discount
Problems involving discount

Challenge: A department store advertises cameras at "25% off original price." What would you pay for a camera which was originally priced at $60?

I. Procedure

A. Meaning of discount; finding discount

1. Discuss with pupils why stores run "sales."

   Tell pupils that when articles are on sale, the original price, before reduction is made, is called the marked price. What are other names for original price? (regular price, list price)

2. Tell pupils that the amount of the reduction is known as the discount.

   How may the reduction or discount be stated? (as a per cent, such as "25% off," "discount of 25%," and so on)

   We call the 25% the rate of discount.

3. Refer to the challenge problem.

   a. What is the original or marked price of the camera? ($60)
   What is the rate of discount? (25%)

   b. Elicit that before we can determine what is to be paid for the camera, we must find the amount of discount. That is to say, we must find 25% of $60.

   c. $\frac{1}{4} \times 60 = \square$ or $.25 \times 60 = \square$.

d. How much is to be paid for the camera? $60 - 15 = \square$

The amount to be paid for the camera is $45.

e. Tell pupils the price actually paid for the camera, $45, is called the sale price, or net price.

f. Have pupils observe that $45 is 75% of $60.

B. Solution of problems

1. Mary bought a dress at a special sale at 10% off the regular price. If the regular price is $25, what is the discount? What is the sale price?

   a. Elicit that to find the amount of discount, we must find 10% of $25. We therefore solve $\frac{1}{10} \times 25 = \square$ or $.10 \times 25 = \square$.

$$\frac{1}{10} \times 25 = 2.5 \text{ or } \begin{array}{r} 25 \\ \underline{\times .10} \\ 2.50 \end{array}$$

   The amount of discount is $2.50.

   b. Elicit that to find the sale price, we subtract $2.50 from $25. $25 - 2.50 = \square$

$$\begin{array}{r} \$25.00 \\ \underline{-\ 2.50} \\ \$22.50 \end{array}$$

   The sale price is $22.50 (which is 90% of $25).

2. During a sale, all merchandise in a furniture store was offered at a discount of 40%. What is the sale price of a sofa marked to sell originally at $360?

   a. Elicit that we must compute the amount of discount before we can find the sale price of the sofa. That is to say, find 40% of $360.

   b. $\frac{2}{5} \times 360 = \square$ or $.40 \times 360 = \square$

$$\frac{2}{5} \times 360 = 144 \text{ or } \begin{array}{r} 360 \\ \underline{\times .40} \\ 144.00 \end{array}$$

   The amount of discount is $144.

c. What is the sale price of the sofa?

3. After several such problems, have pupils generalize that if we know the regular or marked price, and the rate of discount, it is first necessary to compute the amount of discount before we can find the sale or net price.

II. Practice

A. In each of the following exercises, the original price and the rate of discount are given. Compute the amount of discount and the sale price.

	Original Price	Rate of Discount
1.	$16	10%
2.	$2000	6%
3.	$45.95	20%
4.	$220	35%

B. Mr. Lopez bought a car that was originally marked $2980, including all extras, at a reduction of 15%. How much did the car cost him?

C. Robert's father saw two similar bicycles on sale. In one shop, the sign read, "$40 less 20%"; in the other shop, the sign read, "$44 less 25%." Which store made the better offer?

III. Summary

A. What is the meaning of discount?

B. What is the difference in meaning between rate of discount and discount?

C. How is rate of discount usually expressed?

D. How do we compute the amount of discount?

E. When we know the original price and the amount of discount, how do we find the sale price?

F. What new vocabulary did we learn today?

(original price, marked price, rate of discount, discount, sale price)

Lesson 111

Topic: Per cent

Aim: To use the procedure for finding a per cent of a number in problems involving commission

Specific Objectives:

Meaning of commission; finding amount of commission
Problems involving commission

Challenge: Julie sells greeting cards. She receives a commission of 40% on all sales. One week her sales amounted to \$65. How much commission did she receive?

I. Procedure

A. Meaning of commission; finding amount of commission

1. Elicit that some people are paid for their services by receiving a salary or wages. This means they receive the same amount each week or each month. Others, like salesmen, may be paid a certain per cent of the amount of money received for the goods they sell.

   Tell pupils that salesmen are paid a commission for their services.

   a. Why do salesmen sometimes receive salary and commission?

   b. What other people are paid commission for their services? (brokers, real estate agents)

2. Refer to the challenge problem. What is the total amount of sales? What is the rate of commission?

   a. Elicit that to find the amount of Julie's commission, we must find 40% of \$65.

   b. What is the solution of $\frac{2}{5} \times 65 = \square$ or $.40 \times 65 = \square$? (26)

   c. The amount of commission that Julie received is \$26.

B. Solution of problems

1. Mr. Roberts, a real estate agent, sold Mr. Smith's house for

4

$30,000. Mr. Smith agreed to pay Mr. Roberts a commission of 5% for his services. What was the amount of Mr. Roberts' commission?

Elicit that to find the amount of Mr. Roberts' commission, we must find 5% of $30,000.

2. Mr. Edwards, a salesman, receives a weekly salary of $50 plus a commission of 7% of his sales. During one week his total sales amounted to $2685. How much did Mr. Edwards earn that week?

   $187.95 + 50 = \square$.

   a. What was the amount of Mr. Edwards' commission?

   b. Elicit that to find Mr. Edwards' earnings for that week, we must add his weekly salary of $50 to $187.95, the amount of commission he receives.

II. Practice

A. In each of the following exercises, the total sales and the rate of commission are given. Compute the amount of commission.

	Total Sales	Rate of Commission
1.	$85	10%
2.	$950	6%
3.	$45,000	4%
4.	$3500	½%

B. Harry's father receives a weekly salary of $135, plus a commission of 4% of the value of all his sales. One week he sold $1100 worth of merchandise. How much commission did he earn? What was his total salary that week?

C. Mr. Jones paid a 6% commission to a real estate agent for selling his house. The house was sold for $32,000.

   1. How much commission did the real estate agent receive?

   2. How much money did Mr. Jones actually receive from the sale of his house?

D. An employment agency offered Mary a choice of selling jobs. One job paid a straight salary of $85 a week. The other job paid 15% commission on all sales. Which is the better offer if she can be fairly certain of selling at least $600 worth of merchandise each week?

III. Summary

A. What is the meaning of commission?

B. What kind of employees are paid a commission?

C. What is the difference between the rate of commission and the amount of commission?

D. How do we compute the amount of commission?

E. What new vocabulary have you learned today?

(commission, rate of commission)

Lesson 112

Topic: Per cent

Aim: To learn how to find what per cent one number is of another

Specific Objectives:

Per cent names for fractional numbers such as thirds, eighths, and so on
Solving problems which require finding what per cent one number is of another

Challenge: A visiting baseball team won 1 game out of the 3 games played with the home team. What per cent of the games played did the visiting baseball team win?

I. Procedure

A. Finding per cent names for fractional numbers

1. Refer to the challenge problem. Elicit that the problem is based on the number of games played.

   a. What is the ratio of the number of games won to the number of games played?

   b. How can we change the fractional form of a number to the per cent form? (Use the multiplicative identity to obtain a denominator of 100. For example, $\frac{1}{4} = \frac{25}{100} = 25\%$.)

   c. Elicit that this method is not convenient for the fractional number $\frac{1}{3}$.

   d. Guide pupils to see that if we can rename $\frac{1}{3}$ as a decimal in hundredths, we would then be able to rename the decimal as a per cent.

   e. Elicit that a number in fractional form may be changed to decimal form by division as follows:

   $$\frac{1}{3} \text{ or } 3\overset{.33\frac{1}{3}}{\overline{)1.00}}$$

   f. How can you now rename $.33\frac{1}{3}$ as a per cent?

   g. Answer the challenge question.

2. Using similar procedures, have pupils express each of the following ratios as per cents.

   a. $\frac{2}{3}$
   b. $\frac{1}{8}$
   c. $\frac{3}{8}$
   d. $\frac{1}{6}$
   e. $\frac{1}{9}$
   f. $\frac{4}{9}$
   g. $\frac{1}{7}$
   h. $\frac{5}{7}$
   i. $\frac{11}{12}$

3. When is it advisable to use division to rename a fraction as a per cent?

B. Solution of problems which require finding what per cent one number is of another

1. A salesman sold 20 ties one morning. If 13 of these ties were bow ties, what per cent of the ties sold were bow ties?

   a. Elicit that since the problem is based on the number of ties sold,

      1) we first wish to determine the ratio of the number of bow ties sold to the total number of ties sold

      2) we then wish to express this ratio as a per cent.

   b. What is the ratio of the number of bow ties sold to the total number of ties sold?

      $(\frac{13}{20})$

   c. How shall we rename $\frac{13}{20}$ so that it can then be easily expressed as a per cent?

      $\frac{13}{20} \times \frac{5}{5} = \frac{65}{100}$

   d. What per cent of the ties sold were bow ties?

2. Mary bought a coat at a sale for $39.98. The coat originally cost $55.98. What per cent of the original cost was the sale price?

   a. Which number is the problem based upon?

   b. What is the ratio of the sale price to the original cost? (Have pupils round off $39.98 to $40 and $55.98 to $56.)

c. How shall we rename $\frac{40}{56}$ so that it can easily be expressed as a per cent?

d. What per cent of the original cost is the sale price?

3. Elicit that to find what per cent one number is of another, we express the relationship as a ratio, and then rename the ratio as a per cent.

II. Practice

A. Express each of the following ratios as per cents:

1. $\frac{5}{8}$ 2. $\frac{7}{8}$ 3. $\frac{1}{6}$ 4. $\frac{3}{7}$ 5. $\frac{11}{17}$

B. Compute each of the following:

1. What per cent of 50 is 30?
2. 7 is what per cent of 9?
3. 35 is what per cent of 135?
4. What per cent of 150 is 125?
5. 92.3 is what per cent of 130?
6. What per cent of 20 is 25?
7. 4 is what per cent of 3?

C. On a mathematics test, Paul had 14 answers correct out of 16 problems. What per cent did he work correctly?

D. Out of 280 machine parts, 14 were found to be defective. What per cent of the machine parts were found to be defective?

E. There are 30 pupils on register in a class. Today one pupil is absent. What is the per cent of attendance?

F. The Apollo Theatre seats 960 people. For one performance, 880 tickets were sold. What per cent of the seats will be occupied at this performance?

G. Robert bought an overcoat at a sale for $68.95. The coat originally cost $79.95. What per cent of the original cost was the sale price?

III. Summary

A. When is it advisable to change a fraction to decimal form in order to rename the fraction as a per cent?

B. What procedure do we follow when we want to find what per cent one number is of another?

Lesson 113

Topic: Application of Per cent

Aim: To extend the application of per cent to the solution of problems which involve finding a rate

Specific Objectives:

Solving problems involving finding rate of discount
Solving problems involving finding rate of commission

Challenge: At a sale, Joan bought a hat that was marked down to $3 from its regular price of $4. What was the rate of discount?

I. Procedure

A. Finding rate of discount

1. Have pupils recall the following:

   a. The rate of discount is a ratio which compares the amount of discount to the original price.

   b. The rate of discount is expressed as a per cent

2. Refer to the challenge problem.

   a. What is the amount of discount? What is the regular price?

   b. What is the ratio of the amount of discount to the regular price? ($\frac{1}{4}$)

   c. How do we rename the ratio $\frac{1}{4}$ as a per cent? (25%)

   d. Answer the challenge question.

3. After several such problems, elicit that to find the rate of discount we form a ratio of the amount of discount to the regular price and then rename the ratio as a per cent.

B. Finding rate of commission

1. Dennis sold $25 worth of magazine subscriptions. His commission was $10. What was his rate of commission?

   a. Elicit that to find the rate of commission, we must form a ratio.

b. What two amounts in the problem should be compared, and in what order, to form the ratio? (the amount of commission to the total amount of sales)

c. What is the ratio of the amount of commission to the total sales?

d. What per cent names this ratio?

e. Compute the rate of commission as follows: $\frac{10}{25} = \frac{2}{5}$ or 40%.

2. After several such problems, elicit that to find the rate of commission, we form a ratio of the amount of commission to the total sales and then rename the ratio as a per cent.

II. Practice

A. At a sale, Nancy can save \$2 on shoes that usually sell for \$12. What per cent of the regular price is the discount?

B. Helen paid \$1.80 for a book. The regular price was \$2.40. How much did she save? What was the rate of discount?

C. Mr. Stevenson saved \$23.85 in buying a \$79.50 suit at a special annual sale. What was the rate of discount?

D. Mr. Downing receives a commission of \$48 on each TV set which he sells for \$480. What per cent of the price of each TV set is his commission?

E. A real estate broker receives \$1200 commission for the sale of a \$20,000 home. What is his rate of commission?

F. An automobile salesman earned \$75 commission for selling a used automobile for \$500. What was the rate of commission on the sale?

III. Summary

A. What two quantities are compared to find the rate of discount, and in what order?

B. How do you find the rate of discount?

C. What two quantities are compared to find the rate of commission, and in what order?

D. How do you find the rate of commission?

Lesson 114

Topic: Application of Per cent

Aim: To extend the application of per cent to the solution of problems involving profit and loss

Specific Objectives:

Meaning of profit; loss; selling price
Solution of problems involving computation of profit or loss, and selling price when per cent of profit or loss is given
Solution of problems involving computation of rate of profit or loss

Challenge: The school store bought notebooks at 20¢ each and sold each one at a profit of 10% of the cost. What was the selling price of each notebook?

I. Procedure

A. Meaning of profit; loss; selling price

1. Have pupils discuss the uses to which money received by a storekeeper from the sale of his merchandise is put. This money is used to pay for the cost of the merchandise, for rent, advertising, salary, and so on.

   Elicit that the difference between the amount of money that the storekeeper receives for an article he sells, and the amount of money he has paid for the article, is called the profit. It is necessary for the storekeeper to show a profit if he is to continue in business.

2. Why does a storekeeper sometimes sell an article for less than it cost him? (clearance, damaged, storeworn) In such a case, the storekeeper has taken a loss on his sale.

3. Reinforce the meaning of profit; loss; selling price.

   a. A school store sold a notebook for 22¢. The cost of the notebook was 20¢. How much profit was made on the sale?

   b. A store sold a shopworn typewriter for $50. The cost of the typewriter was $60. How much did the storekeeper lose on the sale?

c. A merchant wishes to sell a sweater which cost him \$13.50 at a profit of \$3.50. What should he charge for the sweater?

d. On Saturday night, a supermarket wanted to sell out all leftover lettuce. Each head of lettuce costs the market 19¢. If the market takes a loss of 8¢ on each head of lettuce, what is the selling price?

B. Problems involving computation of profit or loss,and selling price

1. Refer to the challenge problem.

   a. How is the profit expressed? (as a rate)

   b. Elicit that to find the amount of profit, we compute as follows:

   $\frac{1}{10} \times 20 = 2$ or $\begin{array}{r} 20 \\ \underline{\times .10} \\ 2.00 \end{array}$

   The amount of profit is 2¢.

   c. The selling price is 20+2 or 22¢.

2. Because of water damage to some shirts in his store, a merchant decides to take a 25% loss on these shirts. What is the selling price of a shirt which cost him \$4?

   a. What is the rate of loss?

   b. What is the amount of loss on each shirt?

   c. What is the selling price?

C. Problems involving computation of rate of profit or loss

1. A school cafeteria sells ice cream which is bought at 6¢ a cup and sold for 8¢. Find the per cent of profit based on the cost.

   a. What is the profit on each cup of ice cream?

   b. What is the ratio of the profit to the cost of each cup? Recall that this ratio is called rate of profit.

c. What is the per cent name of this ratio?
Thus, the rate of profit based on the cost is $33\frac{1}{3}\%$.

2. In a similar manner, develop the procedure for computing rate of loss.

II. Practice

A. Replace the frames.

	Original Cost	Profit or Loss		Selling Price
1.	$10	☐		$15
2.	$28		$4.50	☐
3.	$2.98	$ .33		☐
4.	☐	$8		$10.50
5.	$14.25		☐	$9.95

B. Umbrellas cost a department store $5.70 each. In order to cover all expenses, as well as make a reasonable profit, the rate of profit is set at 40% of the cost. What is the profit? What is the selling price of each umbrella?

C. Charles bought a baseball glove for $10. A short time later, he received a similar glove as a birthday gift. Since he could not return the one he bought, he decided to sell it to his friend for $8. What was his loss? What was his rate of loss based on the cost?

D. A grocer buys cheese at 55¢ a pound and sells it at 69¢ a pound. What profit does he make on a loaf of cheese weighing 20 pounds?

E. Bill bought a second-hand bicycle for $12 and spent $8 to have it repaired and painted. He then sold it for $22.50. What was the total cost? What profit did he make on the total cost? What was the per cent of profit based on total cost?

F. A storekeeper bought bicycles at $35 each. He adds $10 to the cost of each bicycle for overhead expenses. (Tell pupils that expenses of operating a business, such as rent, light, wages, are called overhead expenses.)

1. What is the total cost of each bicycle to the storekeeper?

2. If he wishes to make a profit of 30% of the total cost, at what price should he sell each bicycle?

G. Hats were marked to sell for $5. The storekeeper paid $3.50 for each hat and the overhead expenses amounted to 50¢ a hat. Were these hats sold at a profit or at a loss? How much was the profit or loss?

III. Summary

A. If you know the cost of an article and you know the amount of profit or loss, how do you find the selling price?

B. If you know the cost of an article and you know the selling price, how do you find the amount of profit or loss?

C. What two amounts are compared, and in what order, to compute rate of profit? rate of loss?

D. What new vocabulary did you learn today?

(profit, loss, selling price, overhead, rate of profit, rate of loss)

Lessons 115 and 116

Topic: Application of Per cent

Aim: To extend the use of per cent to the solution of problems involving simple interest

Specific Objectives:

Concept of interest; terminology
Solution of problems involving simple interest

Challenge: Money paid for the use of a house owned by someone else is called rent. What do we call money paid for the use of someone else's money?

I. Procedure

A. Concept of interest; terminology

1. Discuss the various reasons that people may have for borrowing money: to start a business, to buy a home or a car, to send a child to college, and so on.

2. Refer to the challenge. Have pupils realize that a charge is paid for the use of someone else's property. People pay rent for the use of a house owned by someone else. Similarly, when people borrow money, they must not only pay it back at the time agreed upon, but they also pay for the use of the money for that period of time.

3. Tell pupils that the amount of money borrowed is called the principal. The amount of money paid for using the principal for a certain length of time is called the interest.

   When money is deposited in a savings account, the bank is "borrowing" the saver's money, and therefore pays interest on the deposit.

4. What determines how much interest is paid for the use of borrowed money? Elicit that the amount of interest depends upon:

   a. the principal
   b. the rate of interest
   c. the length of time the money is kept

B. Solution of problems

1. Pose problem: How much would it cost Mr. Swanson to borrow \$400 for one year, if he had to pay a charge of 6% of the amount borrowed?

    a. What is the principal? (\$400)

    b. What name can we give to "a charge of 6%"?(rate of interest)

    c. How do we find the amount of interest? (Find 6% of \$400.)

    d. For what length of time did Mr. Swanson borrow the money? (1 year) How does the length of time for which the money was borrowed affect the amount of interest? (The longer the money is kept, the more the interest that must be paid.)

    If Mr. Swanson borrowed the money for 2 years, how much would the interest be? (\$48) for 1½ years? (\$36)

2. Frank's father borrowed \$350 from a friend to make some home repairs. The rate of interest on the loan was 4% a year. If Frank's father keeps the money for ½ year, what is the total amount he will pay back?

    a. What is the principal? (\$350) the rate of interest? (4%)

    b. Elicit that to find the amount of interest for one year, we compute 4% of \$350.

    $$\frac{4}{100} \times 350 = \square \qquad \text{or} \qquad .04 \times 350 = \square$$

    1) The interest for one year is \$14.
    2) The interest for ½ year is ½ x 14 or \$7.

    c. How much money will be paid back? $350 + 7 = \square$
    The amount of money to be paid back is \$357.

3. After several such problems, have pupils formulate a rule for computing simple interest, that is, interest paid on the principal only.

    To find the interest, we multiply the principal by the rate of interest per year. This product is then multiplied by the number of years (or fractional part of a year) for which the money is borrowed.

4. Elicit that the rule for computing interest may be expressed as a formula: i = pxrxt, where

   i represents the number of dollars of interest
   p represents the number of dollars of principal
   r represents the rate of interest per year (annual rate)
   t represents the number of years

   Note: Unless otherwise stated, the rate of interest is the rate per year.

5. What is the annual rate of interest if the annual interest on a principal of \$200 is \$10?

   a. Elicit that we wish to find what per cent the interest for one year is of the principal.

   b. $\frac{10}{200} = \frac{\square}{100}$ What is the solution?

      The annual rate of interest is 5%.

II. Practice

A. Find the interest on \$800 at 5% for 2 years.

Solution:

$$\begin{array}{rl} \$800 & \text{principal} \\ \underline{\times .05} & \text{rate} \\ \$40.00 & \text{interest for 1 year} \\ \underline{\times 2} & \\ \$80.00 & \text{interest for 2 years} \end{array}$$

or

$$\begin{aligned} i &= p \times r \times t \\ &= 800 \times \frac{5}{100} \times 2 \\ &= 80 \end{aligned}$$

\$80 interest

B. Find the interest on:

1. \$750 for 3 years at 4%
2. \$1000 for 1 year at 4½%
3. \$450 for 2 years, 6 months at 5%

C. What is the amount to be repaid when the principal is \$540 and the interest is \$24.75?

D. At the rate of 5% a year, what is the interest and amount repaid on \$1200 borrowed for 2 years? 1½ years? 9 months?

E. What is the amount to be paid back on a loan of \$5000 for 1 year at 6%? at 7%? at 2½%? at ½%?

F. A carpenter wanted to open a cabinet-making shop. He needed $2400 which he borrowed from a friend and kept for 2 years and 6 months. If he paid interest at an annual rate of 3½%, how much did he repay?

G. What is the annual rate of interest if the principal is $700 and the annual interest is $21?

H. What is the annual rate of interest if the principal is $1500 and the interest for 3 years is $90?

III. Summary

A. What is interest?

B. Why is interest charged?

C. What determines how much interest is charged when money is borrowed?

D. How do you compute the amount of interest? the amount to be repaid?

E. How do you compute the annual rate of interest?

F. What new vocabulary have you learned today?

(principal, interest, annual interest, rate of interest)

CHAPTER XI

This chapter contains suggested procedures for helping pupils understand some concepts of graphing. This understanding is developed through reading, interpreting, and constructing the following types of statistical graphs:

multiple bar	rectangle
multiple line	circle

The lessons in this section can be used to help pupils realize that a graph is a picture representation of the correspondence between the elements of two sets. This correspondence exists regardless of what type of graph is constructed.

Pupils discover that a graph presents data in a compact and understandable form. Desired information can be obtained more quickly and more clearly from a well chosen graph than from words or columns of figures.

In the bar graph, the lengths of the bars emphasize the difference in the magnitude of the elements of one of the sets. Although the width of each bar is entirely arbitrary, as is the choice of free space between the bars, it is a matter of general agreement that all the bars in a graph are of the same width and spaced at the same distance from each other. Pupils are guided to see the advantage of using a multiple bar graph to picture the correspondence between a set of objects or non-numerical elements, and two or more sets of numerical data.

In a similar way, these lessons lead the pupils to discover how to use multiple line graphs. They learn the advantages of using this kind of graph to show the correspondence between a set and two or more sets of numerical data. Whereas the bar graphs are generally used to compare numerical data such as the areas of the world's largest lakes, line graphs are generally used to show trends such as changes in temperature from hour to hour.

The procedures developed for rectangle and circle graphs are also designed to show the correspondence between two sets of elements. However, in the case of the rectangle graph, the area of the entire rectangle represents a set and the areas of the various sections represent subsets of this set. This also holds true in the case of the circle graph. The pupils discover that the area of each section of the graph must be in the same ratio to the area of the entire circle as the magnitude of each measure is to the magnitude of the total set with which these measures are compared.

This work on graphs can be used to review concepts and skills involved in rounding off numbers, ratio, and per cent. The skills and concepts developed in the group of lessons on measurement can also be recalled here.

CHAPTER XI

GRAPHS

Lessons 117-122

Note to Teacher: In order to insure smooth progress of the lessons on graphs, it is suggested that where an overhead projector is available, a series of transparencies be prepared. Where such a projector is not available, a series of rexographed sheets should be prepared. The transparencies and/or rexographed sheets may also list the discussion questions for each graph.

Lesson 117

Topic: Bar Graphs

Aim: To develop an understanding of multiple bar graphs

Specific Objectives:

Review of reading and interpretation of bar graphs
Reading and interpretation of multiple bar graphs

Challenge: The New York Times reported the following temperature readings on one day in 1960.

City	High	Low
Juneau, Alaska	65°	35°
Honolulu, Hawaii	88°	73°
New York, N. Y.	71°	55°

1. Justify the statement that at any time of this particular day it was colder in Juneau than it was in Honolulu.
2. In what other way can this data be presented so that we may make such judgments about the data more easily?

I. Procedure

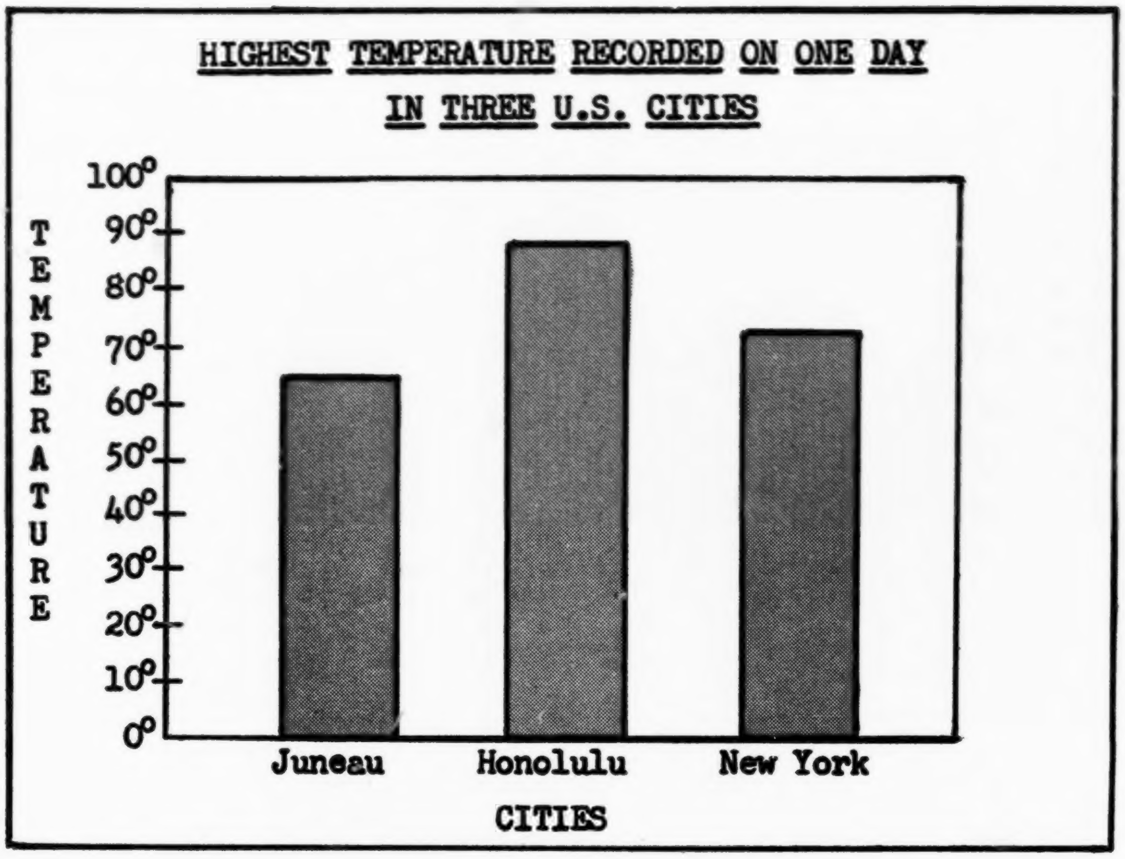
Note: In preparing overhead projector transparencies for this lesson, it is suggested that the bar graphs be so prepared that Graph II may be superimposed on Graph I to form a multiple bar graph.

A. Review reading and interpreting bar graphs

1. Refer to the challenge. Elicit that this data can be presented in the form of a bar graph.

2. Have pupils read and interpret the bar graph shown below.

GRAPH I



a. What is the title of the graph?

b. How are the horizontal and vertical axes labeled?

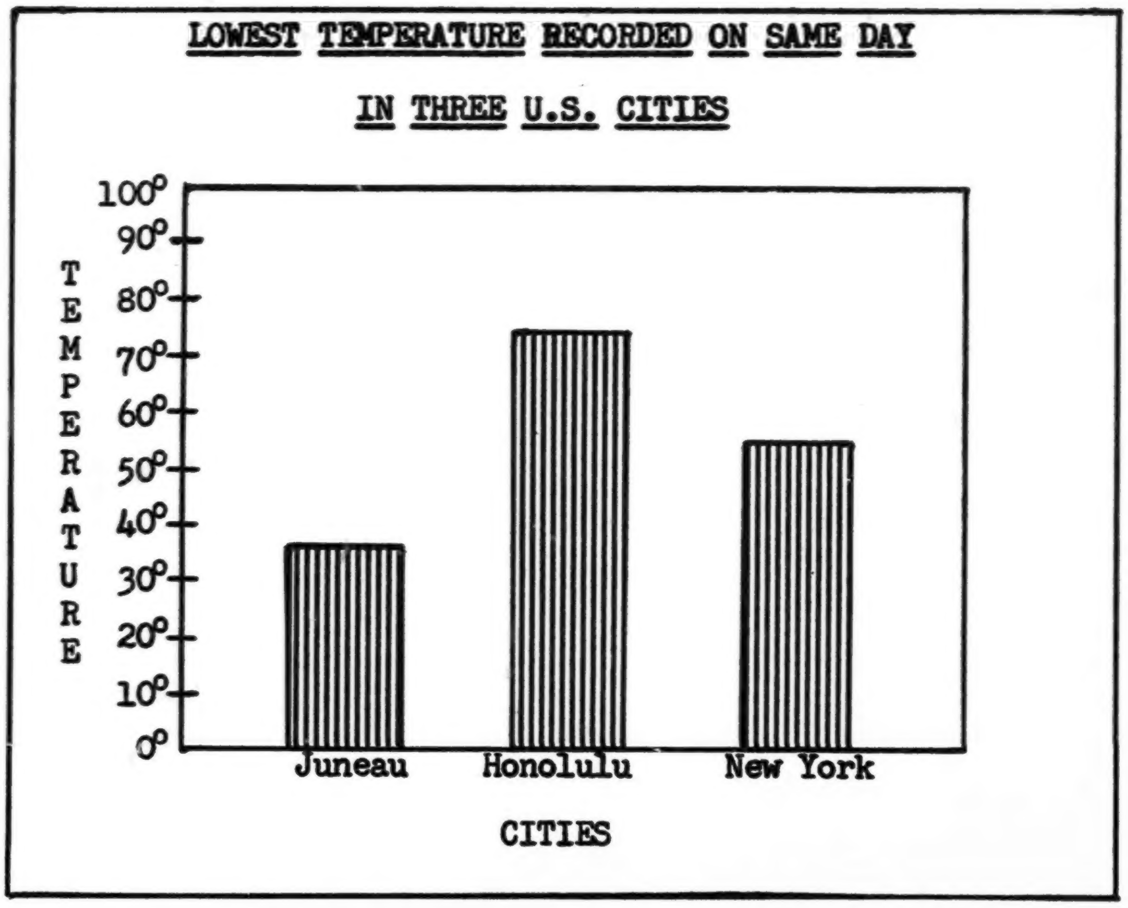
c. What is the scale on the vertical axis?
Where does the scale begin? (at zero)

d. Which city had the highest temperature on that day?
How is this shown on the graph?

e. Between which two cities was there the greater difference in high temperature on that day? How is this shown on this graph?

3. Using similar procedures, discuss the bar graph shown below:

GRAPH II



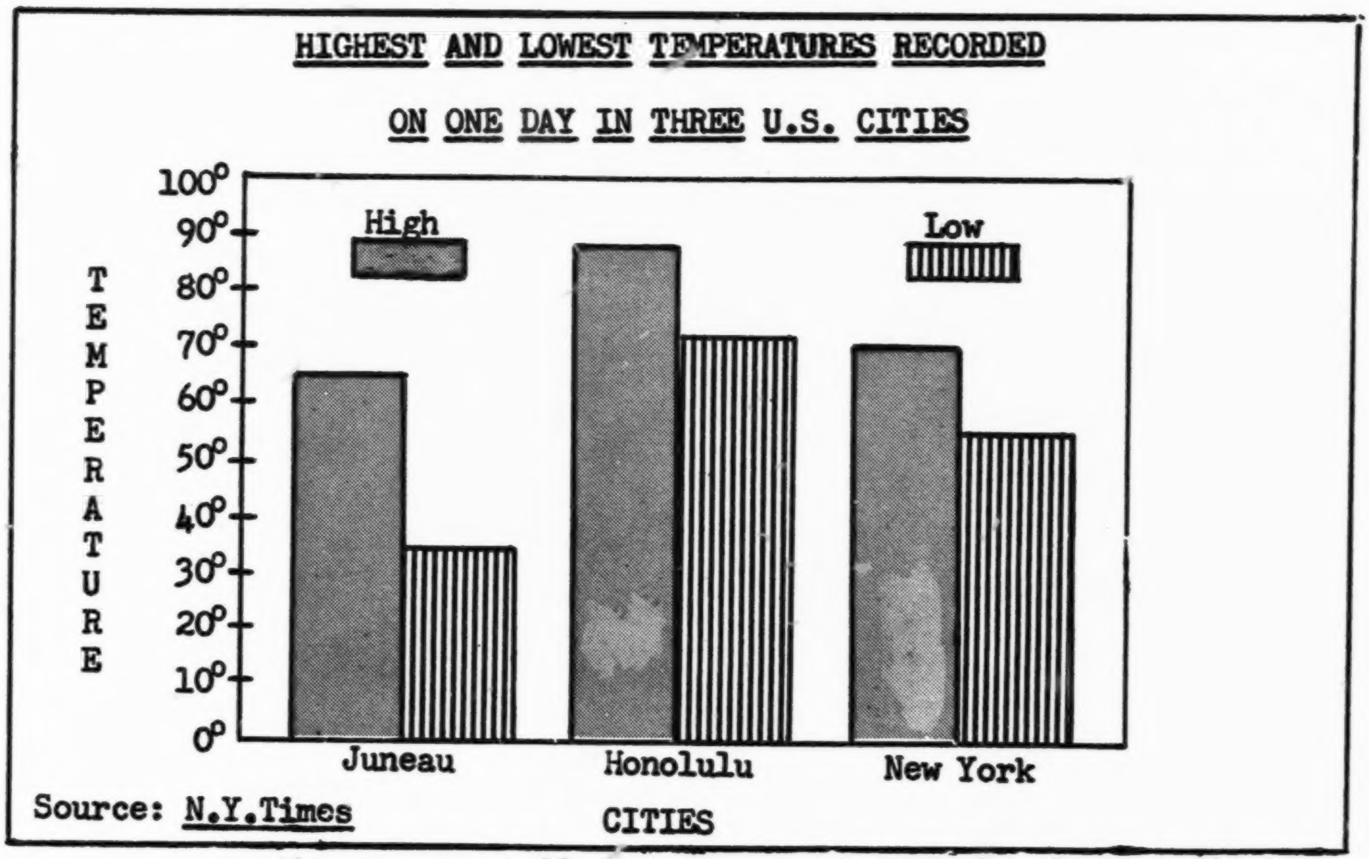
B. Reading and interpreting multiple bar graphs

1. Refer to the challenge question again.
   Look at the two separate bar graphs.
   What other arrangement can be made of the information shown on the two separate bar graphs so that we may compare both sets of facts more easily?

   Lead pupils to see that a single graph may be used showing two bars, one for high temperature, and one for low temperature for each city.

   Tell pupils that such a graph is called a multiple bar graph.

2. Refer to the multiple bar graph representing this data.



HIGHEST AND LOWEST TEMPERATURES RECORDED ON ONE DAY IN THREE U.S. CITIES

Source: N.Y.Times

a. How can you tell from the graph that for Juneau, Alaska one bar represents the high temperature for that day, and the other represents the low temperature for the same day?

b. What was the high and the low temperature for each day in each city?

c. What was the difference between the high and the low temperatures for that day in each city?

d. Answer question #1 in challenge by referring to the graph.

e. Can you justify the statement that at any time of this day it was warmer in Honolulu than it was in New York? Explain.

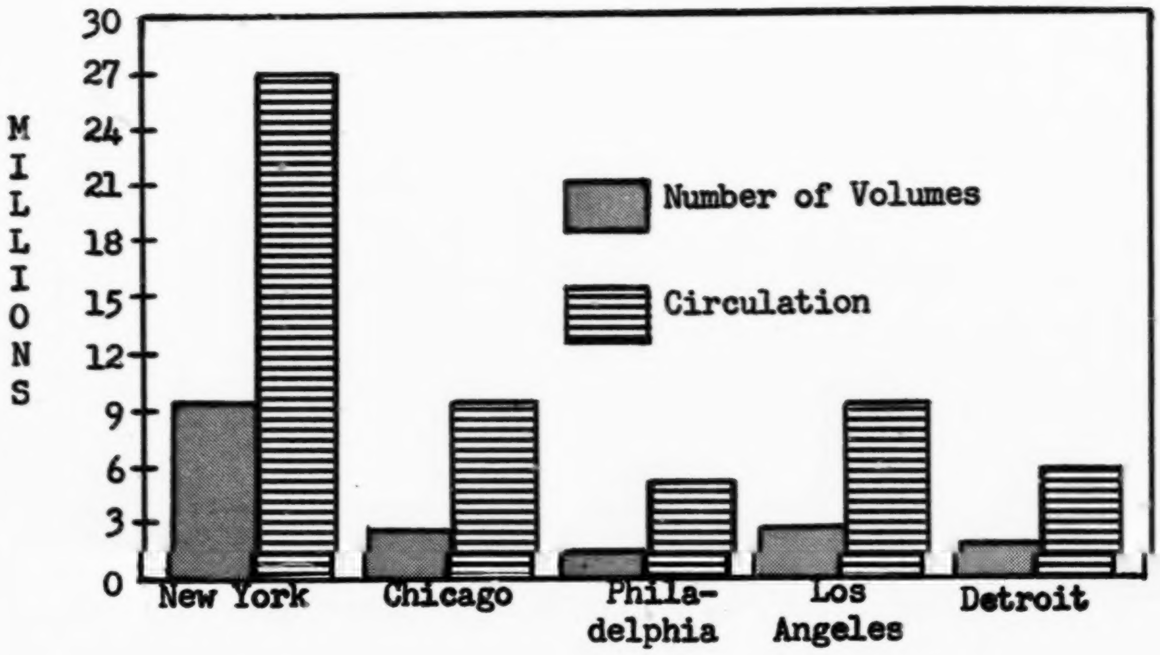
f. Can you tell from the graph which city in the U.S. had the lowest temperature on that day? Why not?

g. When is it advantageous to use a multiple bar graph to represent data?

3. Consider the graph shown below.

PUBLIC LIBRARIES IN AMERICAN CITIES

WITH POPULATION OVER 1,000,000



Source: 1960 World Almanac Book of Facts

a. With what information does this graph deal?
Lead pupils to see that the title of the graph answers this question.

b. What is shown on the horizontal axis? on the vertical axis?

c. What do the differently shaded bars represent?

d. Approximately what was the number of books owned by the public libraries in each city?

e. Which city had the greatest number of books? the smallest number?

f. Approximately what was the number of public library books circulated in each city?

g. Which city had the greatest circulation of books? the smallest?

h. Approximately what was the ratio of the number of books owned to the number of books circulated in each city?

i. For which city was the ratio described above the greatest?

j. Why was a multiple bar graph used to show this data?

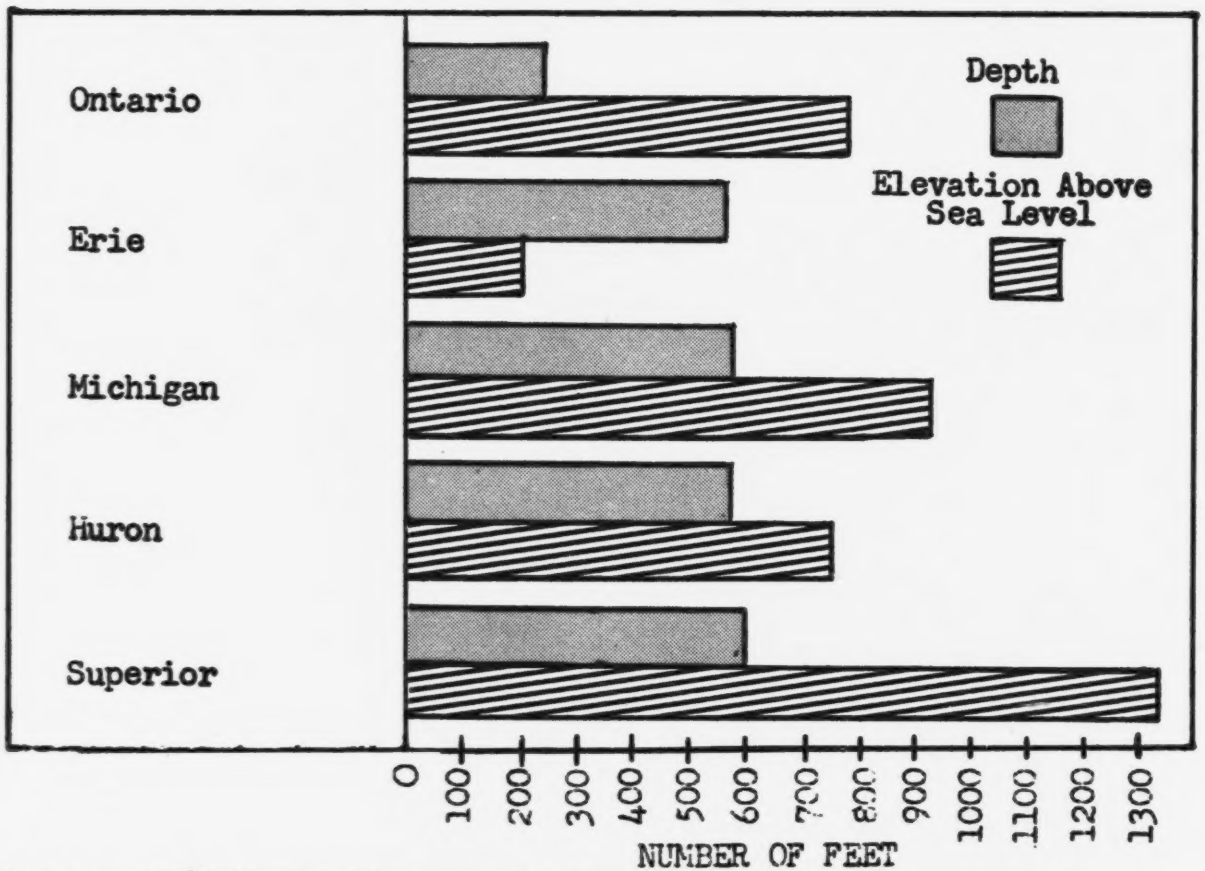
II. Practice

A. The multiple bar graph shown below was pictured in a newspaper to illustrate the following data:

GREAT LAKES:	Maximum Depth (feet)	Elevation Above Sea Level (feet)
Superior	1333	602
Huron	750	581
Michigan	923	581
Erie	210	572
Ontario	778	246

Source: 1960 World Almanac and Book of Facts

GREAT LAKES: THEIR MAXIMUM DEPTH AND ELEVATION ABOVE SEA LEVEL



Source: 1960 World Almanac and Book of Facts

1. How can you tell what this graph is about?

2. How is the vertical axis labeled? the horizontal axis?

3. What scale is used on the horizontal axis?
   Why are the numbers rounded off before the bars are drawn, as, for example, 1333 to 1330; 602 to 600?

4. Could this data have been represented by a vertical bar graph? What was the advantage to the newspaper to use a horizontal bar graph? (to save space)

5. Which Great Lake has the greatest maximum depth?

6. Which Great Lake is at the highest elevation?

7. Which two Great Lakes are at the same elevation?

8. Justify the statement that only Lake Erie has all its water above sea level.

9. If the scale is changed, will the relationships shown by the graph differ? Explain.

10. Can you tell from the graph which of the Great Lakes is the longest? Explain.

B. Have pupils interpret the information that can be obtained from multiple bar graphs collected from newspapers and magazines.

C. (OPTIONAL)

Construct a multiple bar graph to show the following data:

Age in Years	Average Height of Boys	Average Height of Girls
6	44"	42"
9	53"	51"
13	60"	62"
17	69"	63"

1. Draw a vertical and a horizontal axis.
   What shall each axis be labeled?

2. What would be an appropriate scale for the vertical axis?

3. How can we indicate that at each age one bar represents the average height of boys, and the other bar represents the average height of girls?

4. What is an appropriate title for the graph?

5. Why is a multiple bar graph helpful in representing this data?

III. Summary

A. From what item on a graph do you learn what the graph is about? (the title)

B. What determines the length or height of the bar? (the scale used to represent the quantity)

C. Upon what does the choice of a scale depend? (upon the size of the numbers to be represented)

D. With what number must the scale of a bar graph always begin? (zero) Why? (If we do not start at zero, we cannot properly compare the lengths of the bars.)

E. When do you use a multiple bar graph?

F. What new vocabulary have we used today?

(multiple bar graph)

Lesson 118

Topic: Broken-line Graphs (Line Graphs)

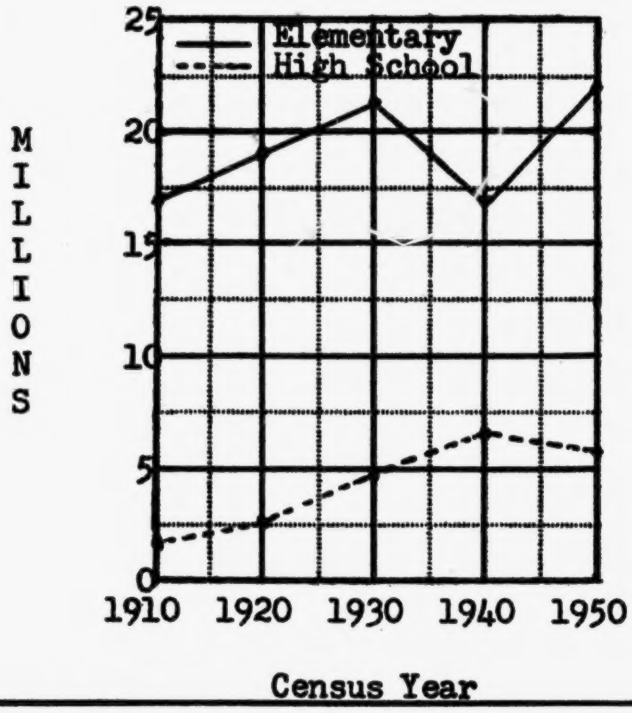
Aim: To develop an understanding of multiple line graphs

Specific Objectives:

Review of reading and interpretation of broken-line graphs
Reading and interpretation of multiple line graphs

Challenge: Consider the following graph:

Number of Elementary and High School Pupils in U.S. from 1910 to 1950



What conclusion can you draw from the data pictured in the graph?

I. Procedure

A. Review reading and interpreting broken-line graphs (also referred so simply as line graphs)

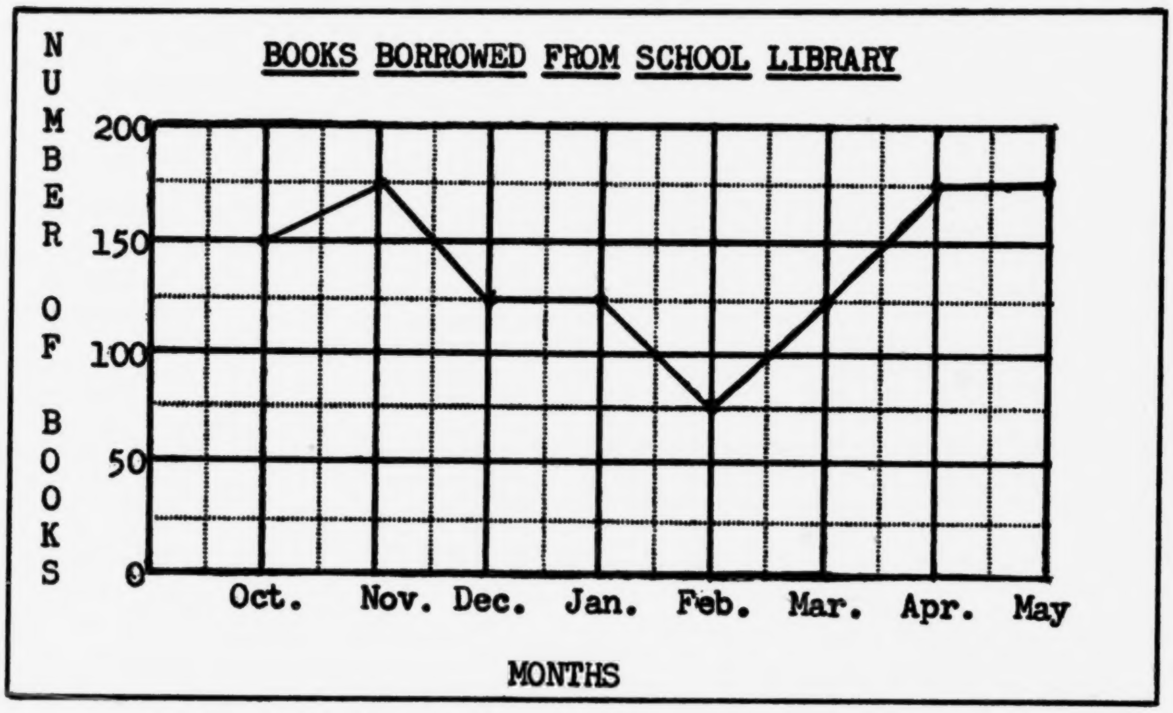
1. Have pupils recall that

a. there are two scales on a line graph, one on the horizontal axis and the other on the vertical axis

b. the scales of a line graph need not start at zero

c. a line graph is useful in showing change or trend

2. Consider the graph shown below.



a. What is the title of the graph?

b. What does the horizontal axis show? the vertical axis?

c. What is the scale on the horizontal axis? on the vertical axis?

d. How many books were borrowed in November? in April?

e. During which months was the greatest number of books borrowed? the same number?

f. For which successive months was there an increase in the number of books borrowed? a decrease?

B. Reading and interpreting multiple line graphs

1. Refer to the graph in the challenge.
What name could be used to describe this type of graph?
Elicit that just as we call a graph with two or more bars for each item a multiple bar graph, so we call a graph with two or more lines a multiple line graph.

2. Refer to the challenge question.
Have pupils see that many interesting conclusions about related data can be drawn from a study of multkple line graphs. For example,

a. There were more elementary school pupils than high school pupils during any one census year.

b. Between the years 1930 and 1940, there was a drop in the elementary school population, while the high school population continued to rise.

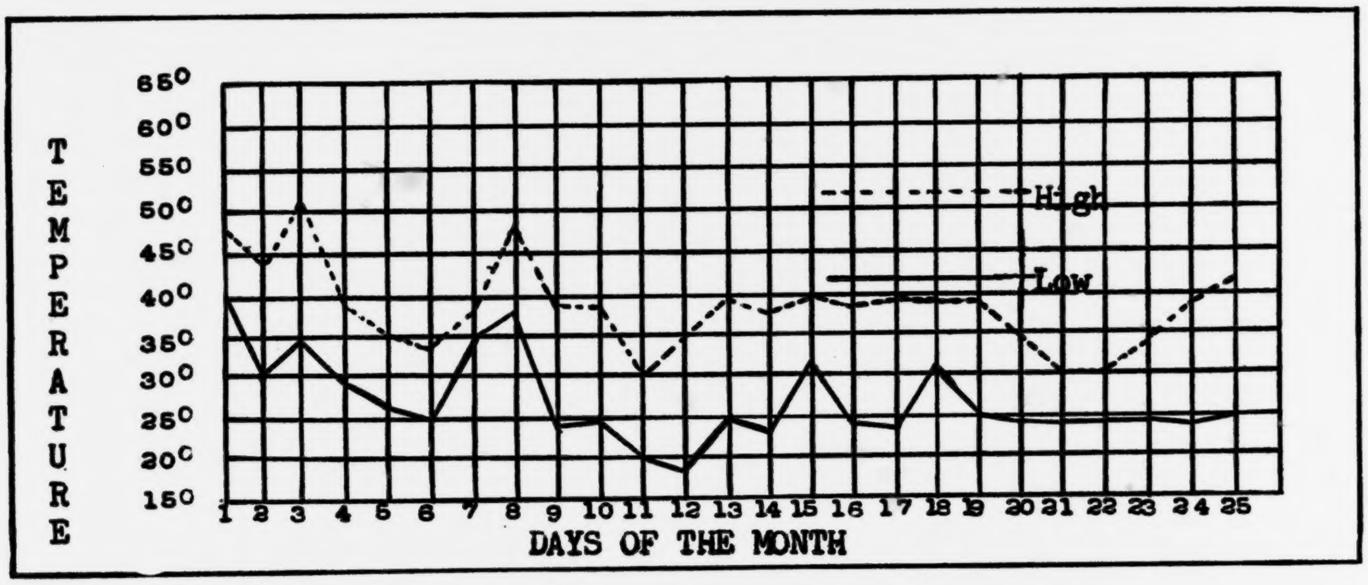
c. Between the years 1910 and 1920, the increase in the elementary school population was greater than the increase in the high school population.

d. Have pupils suggest other comparisons.

3. With respect to the multiple line graph shown, answer the questions below:

NEW YORK'S HIGH AND LOW DAILY TEMPERATURE

IN JANUARY, 1960



a. What is the title of the graph?

b. How are the intervals labeled along the horizontal axis?

c. What scale is used along the vertical axis?

d. What was the high on January 8? the low on January 8?

e. From the graph, on which day was the temperature the highest? the lowest?

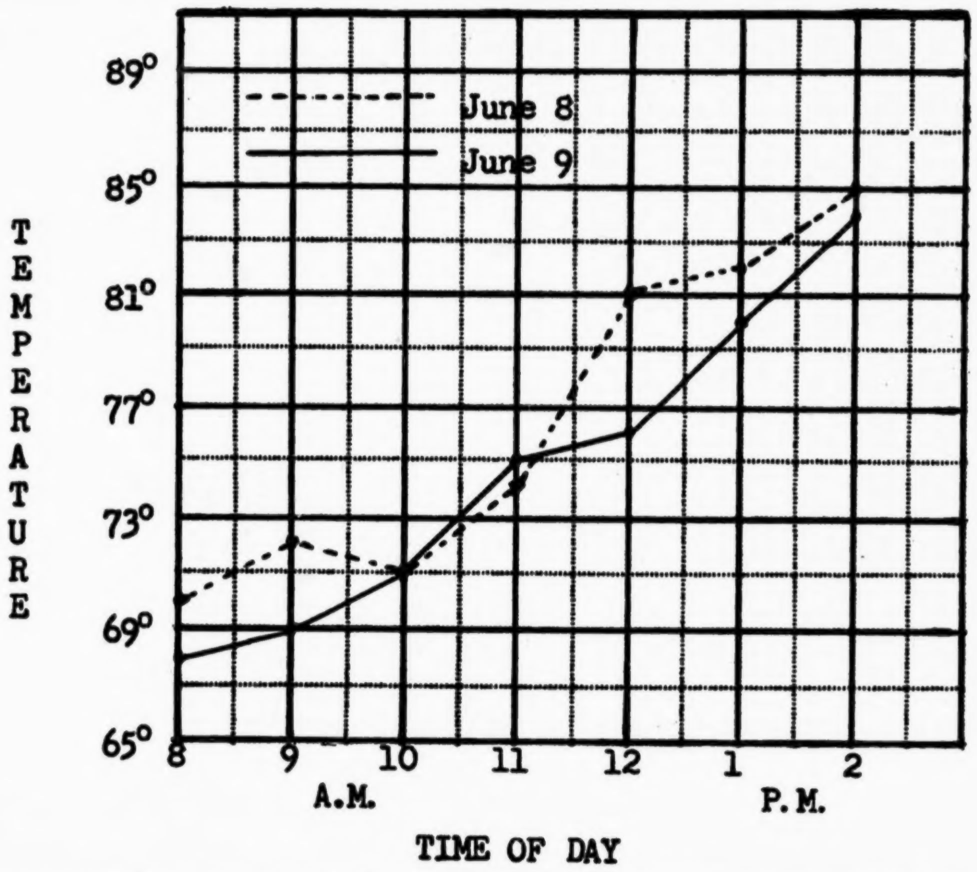
f. On which day does the graph show the least difference in temperature from highest to lowest? (January 7)

g. The average temperature is usually given as the average of the high and low for the day. What was the average temperature on January 3? on January 13?

h. Elicit that some facts cannot be read from the graph. For example, what was the coldest day of the year? sunniest day in January?

II. Practice

A. Consider the multiple line graph shown below:



TEMPERATURE INSIDE ROOM 301 of J.H.S. 000

ON TWO SUCCESSIVE DAYS

1. With what is the graph concerned?

2. How are the intervals labeled along the horizontal axis?

3. What scale is used along the vertical axis?

4. What was the temperature at 11 o'clock on June 8? on June 9?

5. At what time were the temperatures of the two days the same?

6. How much lower was the 8 a.m. temperature on June 9 than on June 10? the 12 o'clock temperature?

7. During which hour on June 8 did the temperature rise the most? (11 a.m. to 12 N)

8. On which of the two days was the temperature generally higher?

9. Why is it permissible to start the scale on the vertical axis with 65°?

B. (OPTIONAL)

Construct a multiple line graph using the following data:

JOHN'S MONTHLY TEST MARKS

IN MATHEMATICS AND SCIENCE

Test	Mathematics (Per cent)	Science (Per cent)
September	75	80
October	80	75
November	80	85
December	90	90
January	85	95
February	100	80
March	100	95
April	90	100
May	95	85
June	80	90

1. What would be an appropriate title for the graph?

2. What would be an appropriate scale on each axis?

3. Draw the graph of the mathematics marks by means of a solid line.

4. Draw the graph of the science marks by means of a dotted line.

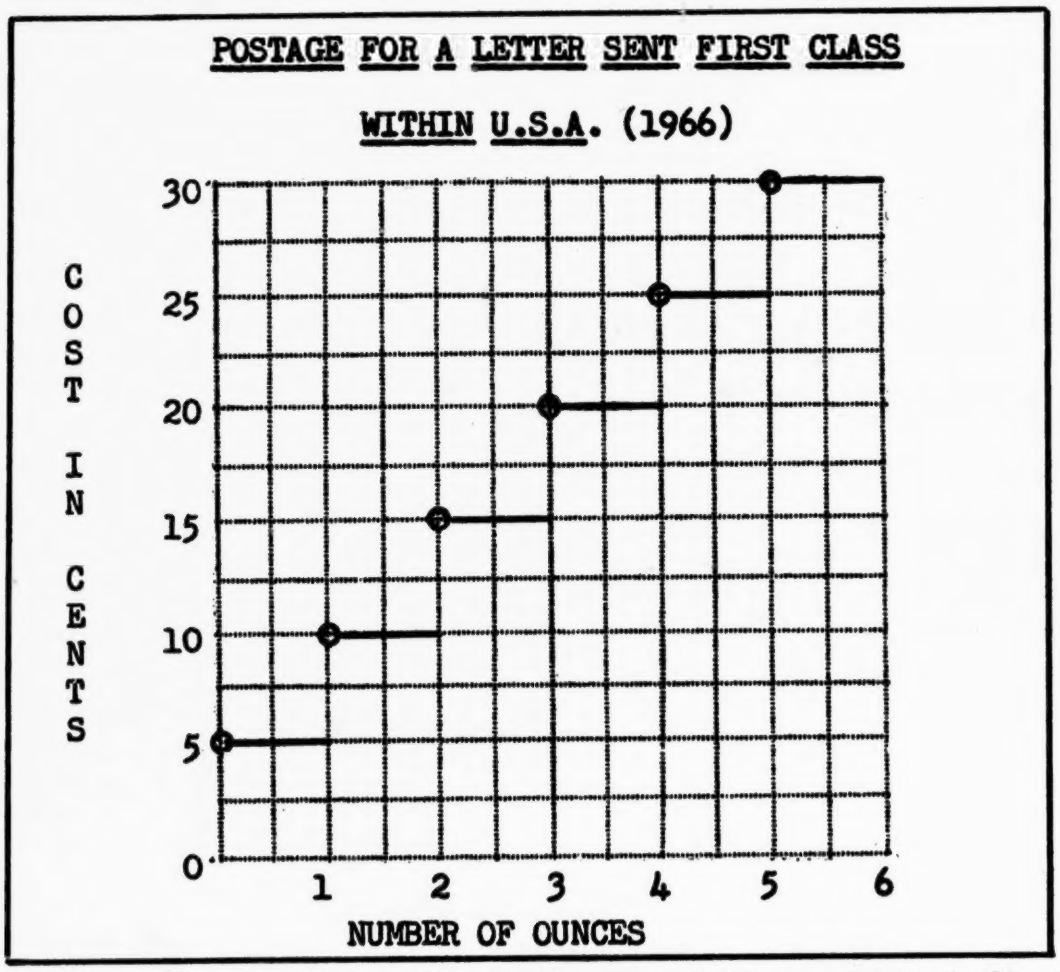
5. In which months did John do better in mathematics than he did in science?

6. In which months did John do better in science than he did in mathematics?

7. Does John prefer mathematics or science? (cannot be answered from the graph)

C. (OPTIONAL)

Have pupils read and interpret the following step graph:



POSTAGE FOR A LETTER SENT FIRST CLASS WITHIN U.S.A. (1966)

Note: The circle on the left end of the segment indicates that this end point is not included in the interval. The dot on the right of each segment indicates that this end point is included.

For example, how much postage would you pay for a letter weighing less than 1 oz.? (5¢); for a letter weighing 1 oz.? (5¢); for a letter weighing 1.5 oz.? (10¢); for a letter weighing 2 oz.? (10¢)

1. How much does it cost to send the following letters first class within the U.S.A.?

   a. $1\frac{7}{8}$ oz.
   b. 3 oz.
   c. $\frac{1}{4}$ oz.
   d. $2\frac{1}{2}$ oz.
   e. $4\frac{3}{4}$ oz.
   f. 4 oz.
   g. $4\frac{1}{4}$ oz.
   h. $5\frac{3}{4}$ oz.

2. Have pupils explain why it is necessary to use the circle at the beginning of each line segment.

III. Summary

A. What are the essential features of every broken-line graph? (title, name of horizontal axis, name of vertical axis, scales on each axis)

B. When is a line graph especially useful?

C. Explain why the scales need not begin at zero on a line graph.

D. When is it advantageous to use a multiple line graph to illustrate data?

E. What new vocabulary have you learned today?

(multiple line graph)

Lessons 119 and 120

Topic: Rectangle Graphs

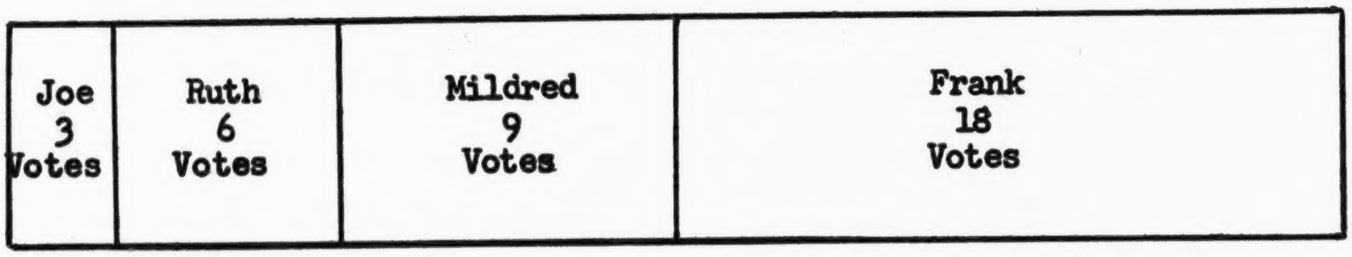
Aim: To develop an understanding of rectangle graphs

Specific Objectives:

Reading and interpreting rectangle graphs
Constructing rectangle graphs

Challenge: Consider the following graph:

ELECTION FOR CLASS PRESIDENT IN CLASS 7-413



How is the appearance of this graph different from that of graphs we have studied so far?

I. Procedure

A. Reading and interpreting rectangle graphs

1. Refer to challenge

a. Elicit that this graph has the shape of a rectangle. Tell class that such a graph is called a rectangle graph.

b. How can you tell what this graph is about? What is the title of the graph?

c. What is the total number of votes cast? (36)

What is the ratio of the number of votes for Joe to the total number of votes? for Ruth to the total number?

Which candidate received $\frac{1}{4}$ of the votes? $\frac{1}{2}$ of the votes?

d. What part of the entire rectangle represents the number of votes for each candidate?

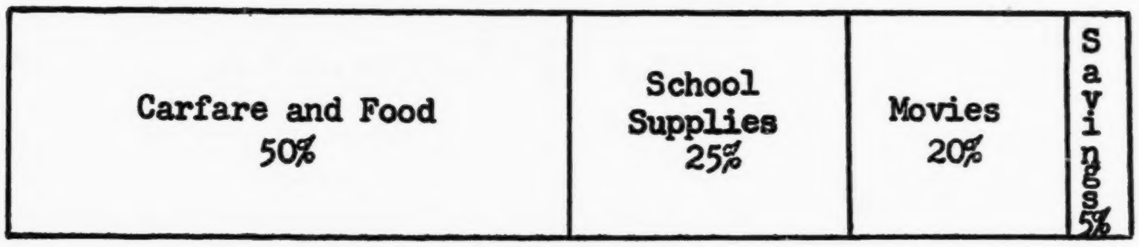
e. What does the whole rectangle represent? (the total number of votes cast)

Guide pupils to see that the size of the section allotted to each candidate depends upon the part he received of the total number of votes.

f. How does a rectangle graph differ from a bar graph? from a broken-line graph?

2. Consider the rectangle graph shown below:

HARRY'S ALLOWANCE OF $4.00



a. What amount of money does the entire rectangle represent?

b. What per cent of Harry's allowance does the entire rectangle represent?

c. What per cent of the allowance was spent on movies? (20%) How much money was spent on movies? (80¢)

d. How much money was used for carfare and food? ($2.00) for school supplies? ($1.00) for savings? (20¢)

e. Why was a rectangle graph used to present this information? Guide pupils to see that rectangle graphs are used to show the relationship of the various parts to the whole (the entire rectangle).

B. Constructing rectangle graphs

1. A football team won 5 games, lost 3 games, and tied 2 games during a season. Show the team's record by means of a rectangle graph.

a. Help pupils decide on a suitable length for the rectangle.

1) What is the ratio of the number of games won to the total number of games played? of the number of games lost to the total number of games played. of the number of games tied to the number of games played?

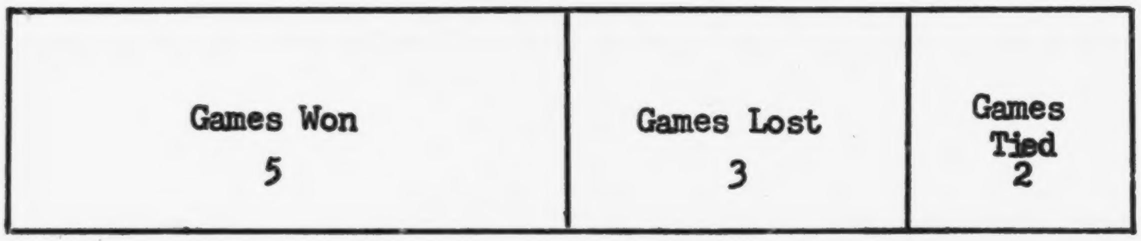
2) What fractional part of the number of games played is the number of games won? ($\frac{5}{10}$ or $\frac{1}{2}$) the number of games lost? ($\frac{3}{10}$) the number of games tied? ($\frac{2}{10}$ or $\frac{1}{5}$)

3) Elicit that although a rectangle of any length may be used, a good choice would be 10" or 5" because of the fractions involved. However, a rectangle of length 10" might be impractical. Why?

b. Help pupils plan the division of the rectangle. If we use a length of 5" for the rectangle, what length shall we use for the games won? ($\frac{1}{2}$ of 5 or $2\frac{1}{2}$") for the games lost? ($\frac{3}{10}$ or 5 or $1\frac{1}{2}$) for the games tied? ($\frac{1}{5}$ of 5 or 1")

c. Have pupils divide the rectangle into the various sections and have them label each section accordingly.

FOOTBALL TEAM'S RECORD



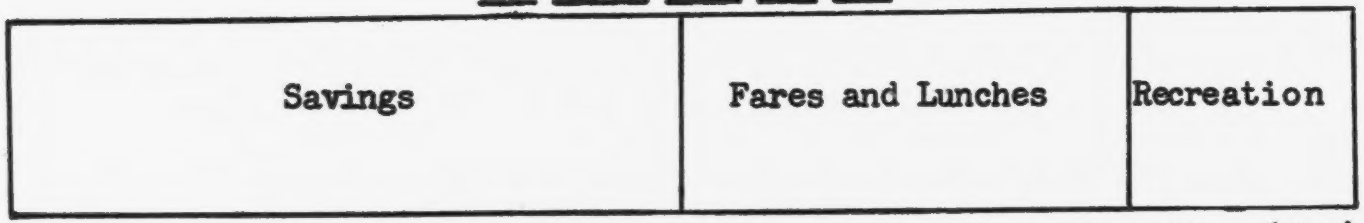
2. David, a high school senior, works in a hardware store after school. He earns $18 a week. He saves 50% of his earnings, spends $33\frac{1}{3}\%$ on fares and lunches, and $16\frac{2}{3}\%$ on recreation. Represent these facts by means of a rectangle graph.

a. Elicit that a rectangle whose length is 6" is a good choice for the graph.

b. How long should the section for savings be made? (50% of 6", or 3") the section for fares and lunches? the section for recreation?

c. Have pupils construct the rectangle graph as follows:

HOW DAVID USES HIS $18

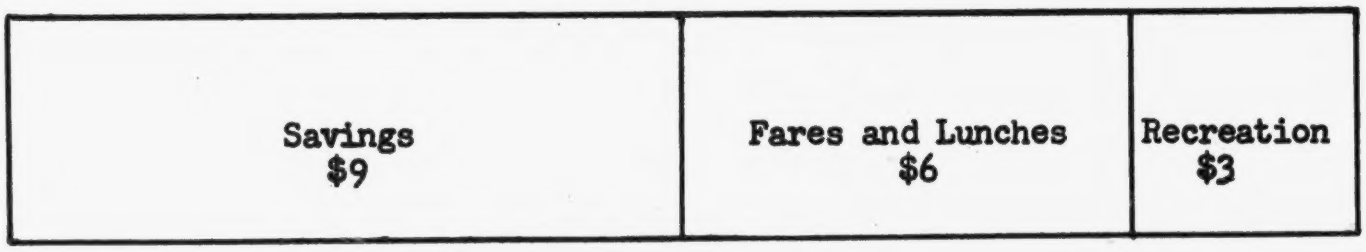


How much money does the entire rectangle represent? ($18)
What per cent of the money does the entire rectangle represent? (100%)

d. How much money does David save?
How much money does he spend for fares and lunches? for recreation?

Have pupils record this information on the graph.

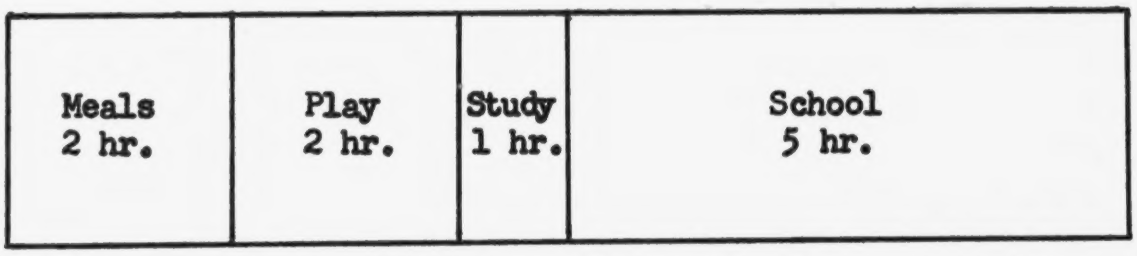
HOW DAVID USES HIS $18



II. Practice

A. Have pupils read and interpret the following rectangle graph:

BEN'S ACTIVITIES



1. How many hours does the entire rectangle represent?
2. What part of the time was spent at school? at play? at study?
3. What per cent of the time was spent at school? at meals? at study?
4. What per cent of the 10 hours does the entire rectangle represent?

B. In his coin collection, Harry had 40 English, 20 French, 10 Japanese, 30 Mexican and 20 Italian coins.

Use a rectangle graph to show what part each kind of coin is of the total collection.

III. Summary

A. What kind of data is most suitably pictured by a rectangle graph? (data showing relationship between the whole and its parts)

B. What are the steps in constructing a rectangle graph?

C. Upon what does the choice of the length of the rectangle depend?

D. What new vocabulary have you learned today?

(rectangle graph)

Lessons 121 and 122

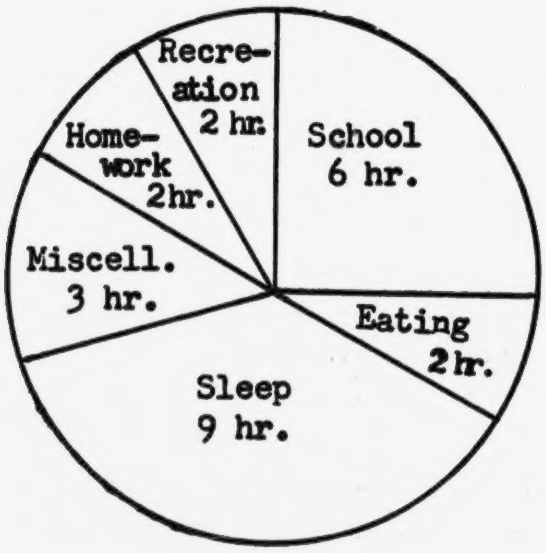
Topic: Circle Graphs

Aim: To develop an understanding of circle graphs

Specific Objectives:

Reading and interpreting circle graphs
Constructing circle graphs

Challenge: HOW JOHN SPENDS A SCHOOL DAY



From the appearance of this graph, what name would you suggest for it?

I. Procedure

A. Reading and interpreting circle graphs

1. Refer to the challenge.
   Guide pupils to see that circle graph would be a suitable name for the graph.

2. How do we know with what a graph is concerned?
   What is the title of this graph?

3. How many hours does the whole graph represent? (24 hours)

   From the graph, estimate the part of the total day that was spent in school; eating; sleeping; recreation; homework; miscellaneous.

On what activity was most of the time spent?

On which activities was the same amount of time spent?

How many times as many hours were spent in sleeping as in eating?

4. If 6 hours were spent in school, what fractional part of the day was spent in school? Compare your answer with size of the section marked "School" on the graph.

   If 9 hours were spent sleeping, what fractional part of the day was spent this way? Compare your answer with the part of the circle marked "Sleep" on the graph.

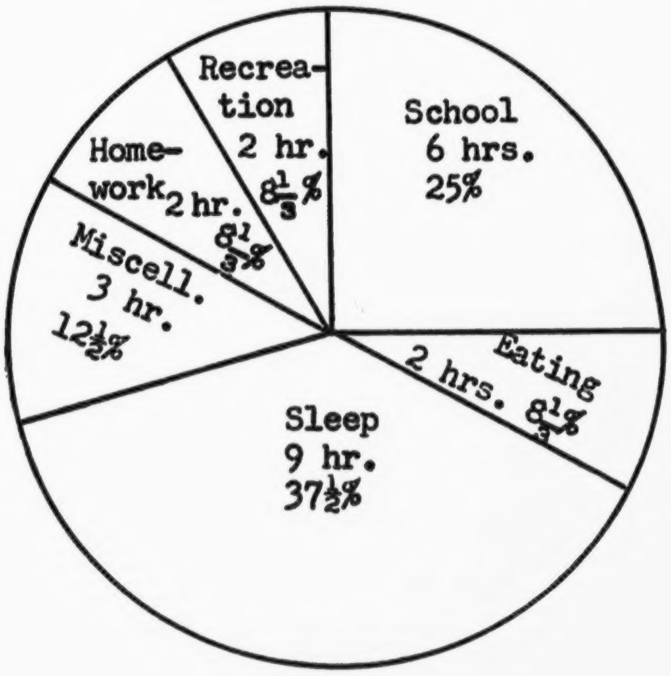
   After similar questions about eating, recreation, homework, and miscellaneous, elicit that the size of the section of the circle graph (called a sector of the circle) allotted to each item is determined by the part of the whole that the item represents.

5. What per cent of the total number of hours in the day is spent on each activity?

   What per cent of the total number of hours does the entire graph represent?

   Indicate these results on the graph as follows:

HOW JOHN SPENDS A SCHOOL DAY



B. Constructing circle graphs

Draw a circle graph to represent the following information: Bill reported that this past year he read 12 fiction books, 12 dealing with science, 2 on travel, 10 in the area of biography and history, and 4 others.

1. What is the total number of books read?

2. What fractional part of 40, the total number of books read, is each type of book?

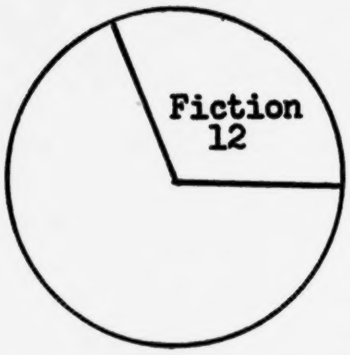
   The results can be tabulated as follows:

Type of Book	Number Read	Fractional Part of 40
Fiction	12	$\frac{12}{40}$ or $\frac{3}{10}$
Science	12	$\frac{12}{40}$ or $\frac{3}{10}$
Travel	2	$\frac{2}{40}$ or $\frac{1}{20}$
Biography	10	$\frac{10}{40}$ or $\frac{1}{10}$
Others	4	$\frac{4}{40}$ or $\frac{1}{10}$
Total	40	$\frac{40}{40}$ or $\frac{20}{20}$ or 1

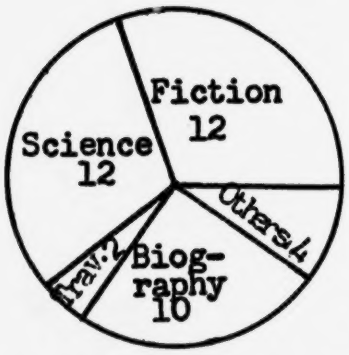
3. Recall that a circle is considered to have 360°. Have pupils determine the number of central angle degrees for each of the sectors into which the circle is to be divided.

Type of Book	Number Read	Fractional Part of 40	Fractional Part of 360	Central Angle
Fiction	12	$\frac{12}{40}$ or $\frac{3}{10}$	$\frac{3}{10} \times 360$	108°
Science	12	$\frac{12}{40}$ or $\frac{3}{10}$	$\frac{3}{10} \times 360$	108°
Travel	2	$\frac{2}{40}$ or $\frac{1}{20}$	$\frac{1}{20} \times 360$	18°
Biography	10	$\frac{10}{40}$ or $\frac{1}{4}$	$\frac{1}{4} \times 360$	90°
Others	4	$\frac{4}{40}$ or $\frac{1}{10}$	$\frac{1}{10} \times 360$	36°
Total	40	$\frac{40}{40}$ or $\frac{20}{20}$ or 1		360°

4. Draw a circle. Indicate the center. Draw a radius. Begin with this radius and mark off in a counter clockwise direction an angle of 108° for the first item, Fiction. Label the section



Continue to mark off the remaining angles; label each part.

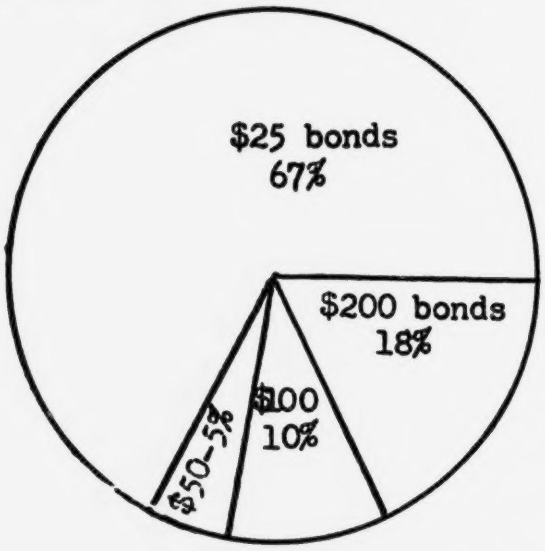


5. What is an appropriate title for the graph? Enter the title to complete the graph.

II. Practice

A. Use the graph below to answer the following questions:

SALE OF U.S. SAVINGS BONDS



1. Which denomination bond was sold in the greatest number? How does the graph show this?

2. Approximately what is the ratio of the number of $25 bonds sold to the number of $50 bonds sold?

3. How does the number of $100 bonds sold compare with the number of larger denomination bonds sold?

4. About what fraction of the total is the number of $50 bonds sold? the number of $100 bonds sold?

5. How many bonds were sold? (cannot tell)

B. Answer each of the following:

1. Choose the correct answer.

   In a circle graph representing 20 books, a sector representing 5 books should have a central angle of: ¼°, 90°, 25°?

2. If the monthly income of a family is $600 and 40% is spent on food, what is the size of the central angle in the sector that represents this item in a circle graph?

3. A circle graph shows that in a school of 60 classes there are 15 classes in the 7th grade. What per cent of the classes are 7th grade classes?

C. Make a circle graph to show the following information taken from the radio page of a daily newspaper.

RADIO PROGRAMS SCHEDULED FOR MONDAY ON ONE STATION

News	3 hours
Music	9 hours
Drama	6 hours
Interview	3 hours
Panel Discussions	3 hours

III. Summary

A. What kind of data is most suitably shown by a circle graph? (data showing relationship between the whole and its parts)

B. What other kind of graph may be used for the same kind of data? (rectangle graph)

C. What is an advantage of using a circle graph?

D. What instruments are used in the construction of circle graphs? (compass, straight edge, protractor)

E. What are the steps in constructing a circle graph?

F. When constructing a circle graph, does the length chosen for the radius affect the relationships? Explain.

G. What new vocabulary have you learned today?

(circle graph, sector)

CHAPTER XII

In the preceding chapters, we have confined our attention to the non-negative rational numbers, i.e., the positive rational numbers and zero. This chapter contains suggested procedures for helping pupils understand the following:

- the need for extending the set of whole numbers to include negative integers
- the concept of set of integers
- the use of the set of integers to describe situations with which the pupils are familiar
- how to associate points on a number line with the set of integers
- order in the set of integers
- addition in the set of integers

Procedures are suggested to guide the pupils to realize that the set of whole numbers is not closed under subtraction. There is, therefore, a need for a set of numbers that has the property of closure under subtraction. By using the pattern approach, the pupils discover that negative integer may be used for the expression "a number less than zero." The set of positive integers, zero, and the set of negative integers are all subsets of the set of integers. This set has the desired property.

In these materials a small raised sign to the left side of the integer is used to indicate the kind of integer represented. Thus, $^{+}5$ is read, "positive five" and $^{-}3$ is read, "negative three." $(^{+}6) + (^{-}4)$ represents the sum of positive six and negative four. In the work of the next grade, a transition to the conventional means of designating directed integers will be made.

Illustrations of the use of integers in everyday situations are presented. The concept of "opposites" is developed. This concept and the fact that every integer is associated with a point on the number line is emphasized. The concept of order is explored.

The geometrical interpretation of integers through the use of a number line is used to develop the operation of addition.

In this grade the only operation with integers that is developed is addition of integers. Pupils learn to add integers using directed line segments which, in effect, is the "vector" approach to addition of integers. A visual approach such as this has a distinct advantage for a first presentation of addition of integers.

CHAPTER XII

THE SET OF INTEGERS

Lessons 123-125

Lesson 123

Topic: Set of Integers

Aim: To extend the number system to include negative numbers

Specific Objectives:

Need for extension of the number system
Concept of integers; of the set of integers
Using positive and negative integers in life situations

Challenge: Consider the set of whole numbers $W = \{0,1,2,3,\ldots\}$.
Is the sum of any two whole numbers always a whole number?
Is the difference of any two whole numbers always a whole number?

I. Procedure

A. Need for extension of the number system

1. Refer to the challenge.

a. Have pupils replace the frames in the following so that true statements result.

$3 + 5 = \square$
$15 + 12 = \square$
$45 + 107 = \square$

$5 - 3 = \square$
$126 - 126 = \square$
$6 - 10 = \square$

Is the sum of any two whole numbers always a whole number?
Is the set of whole numbers closed with respect to addition?

c. Elicit that there is no replacement for the frame in $6 - 10 = \square$ from the set of whole numbers that will result in a true statement. The set of whole numbers is not closed with respect to subtraction.

2. Discuss with pupils the need for a new set of numbers which is closed with respect to subtraction.

B. Concept of integer; of the set of integers

1. Have pupils study the following:

Minuend	5	4	3	2	1
Subtrahend	-3	-3	-3	-3	-3
Difference	$\square$	$\square$	$\square$	$\square$	$\square$

Elicit that the last two examples are "impossible" in the set of whole numbers.

2. Guide the pupils to discover a pattern, namely: from left to right,

the minuend decreases by one
the subtrahend remains the same
the difference decreases by one

3. Have pupils realize that since each difference is one less than the preceding difference, it seems reasonable that the frame in

$$\begin{array}{r} 2 \\ -3 \\ \hline \square \end{array}$$

should be replaced by a numeral which represents "one less than zero"; the frame in

$$\begin{array}{r} 1 \\ -3 \\ \hline \square \end{array}$$

should be replaced by a numeral which represents "two less than zero."

4. Tell pupils that we can represent one less than zero by the symbol $^{-}1$ read as "negative one." In the same way, we represent two less than zero by $^{-}2$, read as "negative two."

We may now write: $\begin{array}{r} 2 \\ -3 \\ \hline ^{-}1 \end{array}$ $\quad$ $\begin{array}{r} 2 \\ -4 \\ \hline ^{-}2 \end{array}$

How would you represent three less than zero? four less than zero?

5. Replace the frames in the following:

$$\begin{array}{r} 8 \\ -6 \\ \hline \square \end{array} \quad \begin{array}{r} 7 \\ -6 \\ \hline \square \end{array} \quad \begin{array}{r} 6 \\ -6 \\ \hline \square \end{array} \quad \begin{array}{r} 5 \\ -6 \\ \hline \square \end{array} \quad \begin{array}{r} 4 \\ -6 \\ \hline \square \end{array} \quad \begin{array}{r} 3 \\ -6 \\ \hline \square \end{array} \quad \begin{array}{r} 2 \\ -6 \\ \hline \square \end{array}$$

6. Tell pupils that numbers such as $^{-}1$, $^{-}2$, $^{-}3$, ..., are called negative integers. They are numbers less than zero. The numbers 1, 2, 3, ..., are called positive integers. These are numbers greater than zero.

7. Have pupils see that we now have a new set of numbers called the set of integers. The set of integers is the union of the set of positive integers, the set of negative integers, and zero. The number zero is neither positive nor negative.

   We can designate the set of integers as

   $$I = \{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}$$

   Elicit that the set of integers is closed with respect to subtraction.

8. Tell pupils that another way of designating positive integers is as follows:

   $$^{+}1, \quad ^{+}2, \; ^{+}3, \quad ^{+}4, \ldots$$

   That is to say, $^{+}1$ names the same number as 1, $^{+}2$ as 2, and so on.

   This method is used when we wish to emphasize the positive nature of a number.

C. Using positive and negative integers in life situations

1. Have pupils see how we may use integers to represent changes, as, for example, the changes in food prices which are shown in the table below

CHANGES IN PRICES OF FOOD ITEMS

Article	Price One Year	Price Next Year	Change
1 gal. gasoline	27¢	30¢	up 3¢, or $^{+}3$
1 lb. butter	79¢	69¢	down 10¢, or $^{-}10$
1 doz. eggs	49¢	63¢	?
1 lb. bread	30¢	32¢	?
1 lb. beef	89¢	89¢	?

2. Have pupils suggest other illustrations of the use of integers to represent changes or variation. (rise and fall in prices of stocks, rise and fall in temperature readings, and so on)

3. Have pupils see how integers may be used to represent opposites.

    a. A boy gained 2 pounds one month and lost 2 pounds the next month.

        1) Elicit that we can think of such a gain and loss as opposites of each other.

        2) If we represent a gain of 2 pounds by $^{+}2$, how can we represent a loss of 2 pounds? ($^{-}2$)

    b. A girl took 5 steps forward, and then 5 steps back. If we represent 5 steps forward as $^{+}5$, how would you represent 5 steps back?

    c. If 25 years from now is represented by $^{+}25$, how would you represent 25 years ago?

    d. If a 5 yard gain in a football game is represented by $^{+}5$, how would you represent a 5 yard loss? a 10 yard gain? a 3 yard loss?

II. Practice

A. Read each of the following:

1. $^{-}4$ (negative four)
2. $^{+}8$
3. $^{-}13$
4. 5

B. Describe the opposite of each of these situations.

1. earning $15
2. a rise in temperature of 7°
3. a 20 yard gain
4. a deposit of $25
5. a profit of $10

C. Use a positive integer to express each of the situations in B. Use an integer to express the opposite of each of these situations.

D. If $^{-}3$ represents a loss of $3, represent a profit of $3.

If an increase of $8 in income is represented by $^{+}8$, how would you represent a decrease of $8 in income? a decrease of $4?

If $^{-}10$ represents a fall in temperature of 10°, how would you represent a rise of 6°?

E. Using integers, represent each of the following situations:

1. jumping 4 feet forward ($^{+}4$ or 4)
2. withdrawing \$10 from the bank ($^{-}10$)
3. losing 3 lbs. in weight ($^{-}3$)
4. fall of 8 points in the stock market ($^{-}8$)
5. earning \$7
6. a debt of \$2
7. 5 games won
8. 15 points below average
9. cost of eggs dropped 3¢ a dozen
10. walking down 6 steps

F. How would you describe $P = \{^{+}1, ^{+}2, ^{+}3, \ldots\}$?

How would you describe $N = \{^{-}1, ^{-}2, ^{-}3, \ldots\}$? (the set of negative integers)

G. Consider $P = \{^{+}1, ^{+}2, ^{+}3, \ldots\}$ and $Z = \{0\}$. What is $N \cup Z$?

H. We call the set which is the union of the set of positive integers and zero the set of non-negative integers.

$$A = \{0, ^{+}1, ^{+}2, ^{+}3, ^{+}4, \ldots\}$$

Why is this set not called the set of positive integers?

I. What name could we use for the set which is the union of the set of negative integers and zero? (non-positive integers)

Why is this set not called the set of negative integers?

III. Summary

A. If we are to have a closure property with respect to subtraction, what kind of number is needed?

B. What name is given to the set of numbers formed by the union of the set of positive integers, zero, and the set of negative integers?

C. Name some subsets of the set of integers.

D. How do you read: $^{+}7$? $^{-}3$?

E. What new vocabulary have you learned today?

(integer, positive integer, negative integer)

Lesson 124

Topic: Set of Integers

Aim: To learn how to associate elements of the set of integers with points on the number line

Specific Objectives:

Associating integers with points on the number line
Order in the set of integers

Challenge: On a thermometer scale, a reading of 10° above zero is marked by a point located 10 units above the zero mark. Where is the location of the mark that indicates 10° below zero?

I. Procedure

A. Associating integers with points on the number line

1. Refer to the challenge question. Elicit that the mark indicating 10° below zero is 10 units below the zero mark. Thus, temperatures of 10° above zero and 10° below zero are represented by marks which are the same distance from the zero mark, but on opposite sides of it.

2. Which integer could be used to represent 10° above zero? ($^{+}10$) 10° below zero? ($^{-}10$).

   Have pupils recall that $^{+}10$ is considered to be the opposite of $^{-}10$.

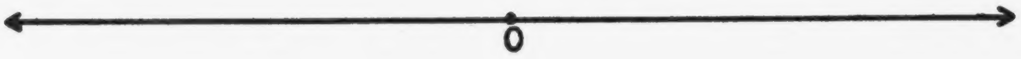
3. Which set of integers would you associate with the whole-number readings which are above zero on a thermometer scale? (set of positive integers) with the whole-number readings which are below zero? (set of negative integers)

4. Have pupils realize that a thermometer scale actually represents a number line in a vertical position. Have them see that a number line in a horizontal position could have positive integers paired with points to the right of zero, and negative integers paired with points to the left of zero.

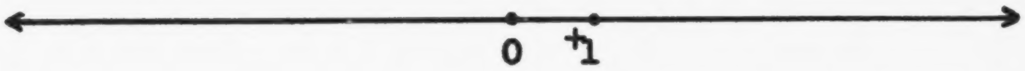
5. Have pupils associate the integers with points on the number

line as follows:

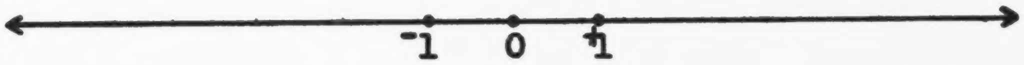
a. Draw a picture of a line in horizontal position and mark a point. Label it zero.



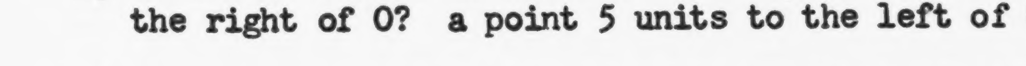
b. Choose a unit length and mark off one unit length to the right of zero. Assign $^{+}1$ to the point which is 1 unit to the right of zero.



c. Mark off one unit length to the left of zero (the same distance but opposite in direction). Since $^{-}1$ is the opposite of $^{+}1$, we will assign $^{-}1$ to the point which is 1 unit to the left of zero.



d. In a similar way, have pupils associate pairs of opposites, such as $^{+}2$ and $^{-}2$, $^{+}3$ and $^{-}3$, with points on the number line.



Emphasize that each of the numbers in a pair of opposites are the same distance from 0, but on opposite sides of 0.

e. Pose questions such as:

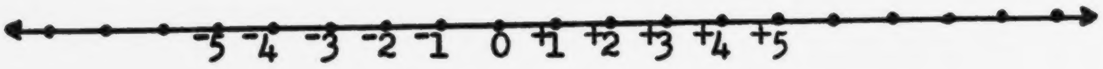
1) With what number do we associate a point 5 units to the right of 0? a point 5 units to the left of 0?

2) Where is the point that is paired with $^{-}8$ located? the point that is paired with $^{+}8$?

6. Elicit that there is a one-to-one correspondence between the set of integers and some of the points on the number line. Whereas every integer may be paired with a point on the number line, not every point on the number line corresponds to an integer.

B. Order in the set of integers

1. Consider the number line below.



a. Of the two numbers, 0 and $^+1$, which is the greater? ($^+1 > 0$)

How is the point associated with $^+1$ located with respect to the point associated with 0? (The point associated with $^+1$ is to the right of the point associated with 0.)

b. Of the two numbers, $^-1$ and 0, which is the greater?

How is the point associated with $^-1$ on the number line located with respect to the point associated with 0?

2. After several such illustrations, as, for example, $^-2$ and $^-1$, $^+4$ and $^+3$, elicit that of two integers, the greater number is associated with a point on the number line which is to the right of the point associated with the smaller. That is to say, the graph of the greater number is to the right of the graph of the lesser.

3. Judge the following sentences true or false. Refer to the number line if necessary.

a. $^-5 > ^-6$

b. $^-7 > ^-3$

c. $0 < ^-4$

d. $^-6 > ^+2$

e. $^-1 > ^-10$

f. $^-1 > ^+2$

4. Have pupils realize that every negative integer is less than any positive integer.

II. Practice

A. Interpret each of the following as a thermometer reading above or below zero.

1. $^-5$ 2. $^-3$ 3. $^+32$ 4. $^+90$ 5. $^-55$

B. Arrange the following integers in the order in which their graphs appear on the number line (from left to right):

1. $4, {}^-3, 0, 3, {}^-1, 6, {}^-4, 11$

2. ${}^-1, {}^-7, {}^-10, {}^-5, {}^-11, {}^-3, {}^-9, 0$

3. ${}^+8, {}^+3, 5, {}^+7, {}^+1, {}^+10, {}^+12$

C. Is each of these statements true? If not, make each a true statement.

1. ${}^+6 > {}^+4$

2. $0 < {}^-1$

3. $0 < {}^+1$

4. $0 > {}^-6$

5. ${}^-3 > {}^-2$

6. ${}^-5 < {}^-8$

7. ${}^+3 > {}^-5$

8. ${}^-100 < 0$

D. For each of the following pairs of integers, use either > or < to write a true sentence about the integers:

1. ${}^+6, {}^+8$

2. $7, 3$

3. $0, {}^-2$

4. $1, {}^-1$

5. ${}^-5, {}^-3$

6. ${}^-4, {}^-7$

7. ${}^-2, 1$

8. ${}^-11, {}^-6$

E. Match the descriptions in Column I with the sets designated in Column II.

Column I	Column II
Set of integers which are:	
1. greater than 3 and less than 6	$A = \{{}^-3\}$
2. greater than ${}^-1$ and less than ${}^+4$	$B = \{{}^-2, {}^-1, 0\}$
3. less than ${}^+1$ and greater than ${}^-3$	$C = \{\ \}$
4. greater than ${}^-4$ and less than ${}^-2$	$D = \{4, 5\}$
5. greater than ${}^-5$ and less than ${}^-4$	$E = \{0, {}^+1, {}^+2, {}^+3\}$

III. Summary

A. Show how the elements of the set of integers are associated with points on the number line.

B. How are the graphs of an integer and its opposite located with respect to 0 on the number line?

C. What subset of the set of integers is associated with points to the right of zero on the number line? with points to the left of zero?

D. If one number is greater than another, how do their graphs appear on the number line? (The graph of the greater number is to the right of the graph of the lesser.)

Lesson 125

Topic: Set of Integers

Aim: To learn how to add integers using the number line

Specific Objectives:

Representing addition of integers
Using the number line in finding the sum of integers

Challenge: Replace the frame so that a true statement results.

$$(^+9) + (^-6) = \square$$

Explain your answer.

I. Procedure

A. Representing addition of integers

1. Refer to the challenge.

a. Have pupils recall what a raised "$^+$" or "$^-$" sign placed in front of a numeral indicates.

b. What does $^+9$ mean? What does $^-6$ mean?

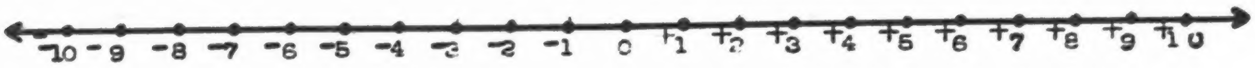
2. What do you think the plus sign between $(^+9)$ and $(^-6)$ in $(^+9) + (^-6)$ means?

Tell pupils that parentheses are used to enclose the integers to avoid confusion of the sign of the numeral with the sign of the operation to be performed.

2. Elicit that $(^+9) + (^-6)$ represents the sum of positive nine and negative six.

B. Addition of integers using the number line

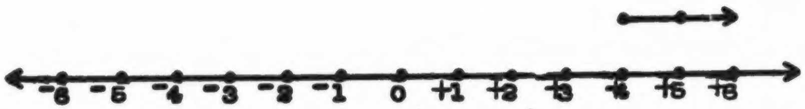
1. Have pupils draw a picture of a number line as shown below.



We shall agree that positive numbers can be represented by moves to the right from any point on the number line, and negative numbers by moves to the left.

a. Find the sum $(^{+}4) + (^{+}2)$.

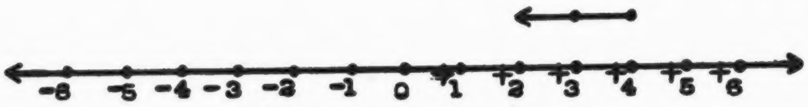
We start at $^{+}4$ and move 2 units to the right. The move ends at $^{+}6$.



$(^{+}4) + (^{+}2) = {}^{+}6$

b. Find the sum $(^{+}4) + (^{-}2)$.

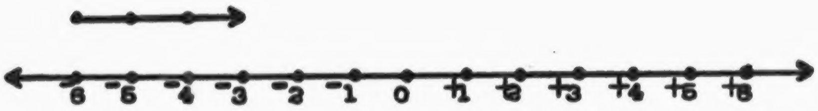
We start at $^{+}4$ and move 2 units to the left. The move ends at $^{+}2$.



$(^{+}4) + (^{-}2) = {}^{+}2$

c. Find the sum $(^{-}6) + (^{+}3)$.

We start at $^{-}6$ and move 3 units to the right. The move ends at $^{-}3$.



$(^{-}6) + (^{+}3) = {}^{-}3$

2. Refer to the challenge question again.
Have pupils use a picture of a number line to compute the answer.

3. Using a picture of a number line, compute each of the following:

a. $(^{+}5) + (^{+}4) = \square$

b. $(^{-}3) + (^{-}7) = \square$

c. $(^{-}7) + (^{+}2) = \square$

d. $(^{+}12) + (^{-}13) = \square$

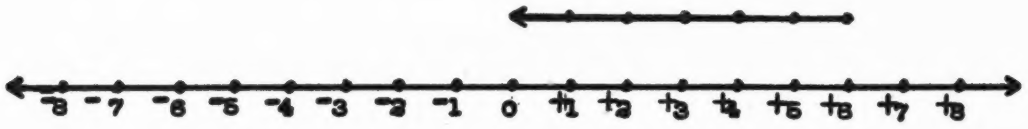
e. $(^{-}5) + (^{+}9) = \square$

4. Use a picture of a number line to investigate the sum of an integer and its opposite.

   a. Find the sum $(^{+}6) + (^{-}6)$.

      1) Where do we start?

      2) How many units and in which direction do we move?



      3) Elicit that $(^{+}6) + (^{-}6) = 0$.

   b. After several such illustrations, have pupils conclude that the sum of an integer and its opposite is zero.

II. Practice

A. Represent each of the following by moves from any point on the number line.

1. $^{+}3$ (3 units to the right)	4. $^{+}12$
2. $^{-}10$ (10 units to the left)	5. $^{-}12$
3. $^{-}8$	6. 0

B. Refer to a picture of a number line to answer each of the following:

   1. If you start at $^{-}6$ and move 7 units to the right, what point will you reach?

   2. If you start at $^{+}3$ and move 5 units to the left, what point will you reach?

C. Using the set of integers, represent each of the following plays of a football team. Then represent the result, using a picture of a number line, if necessary.

   1. a loss of 5 yards followed by a loss of 2 yards
   2. a gain of 8 yards followed by a gain of 1 yard
   3. a loss of 10 yards followed by a gain of 15 yards
   4. a gain of 6 yards followed by a loss of 6 yards
   5. a loss of 8 yards followed by a gain of 5 yards

D. Using the number line, compute each of the following:

1. $(^{+}5) + (^{+}7)$
2. $(^{-}3) + (^{-}5)$
3. $(^{+}12) + (^{-}8)$
4. $(^{-}6) + (^{+}9)$
5. $(^{-}6) + (^{+}1)$
6. $(^{+}7) + (^{-}7)$
7. $(^{+}3) + (0)$
8. $(0) + (^{-}2)$

E. Is each of the following statements true?
Show this by using the number line to compute the result in each case.

1. $(^{+}3) + (^{-}8) = (^{-}8) + (^{-}3)$ 2. $(^{-}9) + (^{-}2) = (^{-}2) + (^{-}9)$

What property of the addition of integers is illustrated?

F. Use the replacement set $S = \{^{-}2, ^{-}1, 0, ^{+}1, ^{+}2\}$, to find the solution set for each of the following:

1. $a + (^{+}3) = (^{+}1)$ 2. $x + (^{-}5) = (^{-}3)$ 3. $(^{+}7) + y = (^{+}7)$

III. Summary

A. How do we distinguish between the sign that tells us whether an integer is positive or negative, and the sign that tells us what operation, such as addition, is to be performed?

B. How can we represent a positive integer as a move on a number line? a negative integer as a move on a number line?

C. Make up a problem that can be represented as the addition of two integers. Show how you would compute the result using a picture of a number line.

INDEX FOR MATHEMATICS: 7th Year

PART I and PART II

Acute angle, 375-378
Addition
  associative property, 50-58, 224-227
  commutative property, 46-49, 54-58 224-227
  of decimals, 289-290, 292
  identity element for, 47
  of integers, 453-456
  of measures, 325, 327, 332
  number line in, 215-217
  problems involving, 59-63
  properties of, 45-59, 223-227
  of rational numbers, 215-227
  related to union of sets, 40-44
  of whole numbers, 50-58
Angle(s)
  acute, 375-378
  adjacent, 379-380
  central, 367-368, 372
  congruent, 368, 381-382
  exterior of, 364
  interior of, 364
  meaning of, 110-111
  measure of, 365-370
  naming, 364
  obtuse, 375-378
  right, 374-378
  side of, 364
  straight, 375-378
  supplementary, 380-383
  vertex of, 364
  vertical, 380-383
Arc, 349-351, 367
Area
  meaning of, 359-360
  of rectangular region, 360-363
  of square region, 360-363
  standard units of measure for, 361
Associative property of addition
  for rational numbers, 224-227
  for whole numbers, 50-58
Associative property of multiplication
  for rational numbers, 172-173
  for whole numbers, 72-75
Bar graph, 416-423
Base(s)
  expressing numbers in various, 27-30
Base-five numeral(s)
  grouping in, 31
  place value of, 32-35
  renaming as base-ten numerals, 34-35
  renaming base-ten numerals as, 37-38
Base-ten numeral(s)
  exponential form, 20-22
  place value of, 16, 23-26
  renaming base-five numerals as, 34-35
  renaming as base-five numerals, 37-38
Braces, 4
Broken-line graph, 424-430

Cardinal number, 7
Central angle, 367-368, 372
Chord of a circle, 348, 350
Circle
  arc of, 349-351, 367
  central angle of, 367-368, 372
  chord of, 348, 350
  circumference of, 352-354
  diameter of, 348-350
  radius of, 347, 350
Circle graph, 435-440
Circumference, of a circle, 352-358
Closed figures, 120-124
Closure property
  for addition of rational numbers, 223
  for addition of whole numbers, 45
  for multiplication of rational numbers, 171
  for multiplication of whole numbers, 67
Common denominator
  finding, 218, 221
  least, 219, 221
Common factor, 158
Common multiple, 180
Commutative property of addition
  for rational numbers, 224-227
  for whole numbers, 46-49

Commutative property of multiplication
  for rational numbers, 172-173
  for whole numbers, 68-71
Comparing rational numbers, 184-188
Complete factorization, 153-154, 156
Composite number, 148-151
Cone, 136-138
Congruent
  angles, 368, 381-382
  line segments, 311-312
Cube, 137-138
Curve, simple closed, 121-124
Cylinder, 136-138

Decimal(s)
  addition of, 289-290, 292
  changing to per cent form, 389-392
  comparing, 278-279
  as divisors, 304-306
  division of, 297-306
  equivalent, 277-279
  expressed as equivalent fractional numeral, 277-280
  expressing rational numbers as, 280-282
  multiplication of, 293-295
  place value in, 275-276
  renaming the dividend, 299-301
  rounding, 285-288
  subtraction, 291-292
Degree, 368-373
Denominator
  common, 218, 221
  least common, 219, 221
Diameter, of a circle, 348-351
Digits
  value of, 16-17
  value of in base-five numeration, 32-35
  value of in decimal, 276-277
Discount, 396-398
Disjoint sets, 40, 42-44
Distributive property of multiplication over addition, 76-82, 231-235
Dividend, 83
Divisibility, 147, 154-157
Division
  of decimals, 297-306
  of measures, 326-328, 333
  by one, 69, 207, 212
  problems involving, 92-96, 212
  of rational numbers, 207-212
  short form of, 84-86
  using reciprocals in, 208-212
  of whole numbers, 83-91
Divisor
  meaning of, 83, 147
  multiplication of dividend and by same number, 208
  rational number as, 208-212

Empty set, 9-12
Endpoint
  of a line segment, 102
  of a ray, 109-113
Equations
  meaning of, 260
  solution sets of, 266-268
  solving, 264-265
Error of measurement, 315
Estimating
  in dividing with decimals, 298
  in multiplying with decimals, 293
  quotients, 87-91
Expanded numerals, 17-19
Exponent(s)
  form of a numeral, 21-22
  meaning of, 20-22
Exterior
  of an angle, 364
  of simple closed curve, 122-124
  of space figure, 136

Factor(s)
  common, 158
  greatest common, 159-161
  meaning of, 146
  prime, 152-157
Factorization, complete, 153-154, 156-157
Figure
  closed, 120-124
  geometric, 100
  plane, 122-124
  simple closed, 121-124
  space, 134-138
Fractional number(s) See Rational number
Fractional numeral
  equivalent, 167
  simplest form of, 175-178
  used to express a rational number, 166

Geometry, 98-138
Graph(s)
  bar, 416-423
  circle, 435-440
  line, 424-430

  - rectangle, 431-434
  - of solution sets, 269-272
- Greatest common factor, 159-161, 200-201
- Greatest possible error of measurement, 316-318

- Hexagon, 127
- Hindu-Arabic numeration system, 16-19

- Identity element
  - for addition, 47-49
  - for multiplication, 69-71, 174-175
- Inch, 309
- Inequalities
  - graphs of, 269-272
  - meaning of, 260
  - solution sets of, 266-268
- Integers, 442-456
- Interest, 411-414
- Interior
  - of an angle, 364
  - of simple closed curve, 122-124
  - of space figure, 136
- Intersection
  - of line and plane, 115, 118
  - of lines, 107-108
  - of planes, 116, 118
  - of sets, 140-145
  - of sets of multiples, 180

- Least common denominator, 219-221
- Least common multiple, 181-183
- Line(s)
  - intersection of, 107-108
  - meaning of, 105
  - naming, 105
  - parallel, 130-131
  - perpendicular, 375-378
  - and planes, 115-116, 118-119
  - and points, 106-108
  - and segments, 105
  - symbol for, 105
- Line graph, 424-430
- Line segments
  - and circles, 347-349
  - congruent, 311
  - endpoints of, 102
  - meaning of, 102
  - measure of, 309-312
  - naming, 103
  - union of, 125
- Linear measure, 309-324

- Mathematical statement, 259
- Measure(s)
  - addition and subtraction of, 325-326
  - of angles, 365-373
  - converting units of, 322-323
  - of line segments, 309-324
  - multiplication and division of, 326-328
  - relations between, 319-322
  - standard units of area, 360
  - standard units of length, 309
- Measurement
  - English system, 3340340
  - error in, 315
  - greatest possible error of, 316-318
  - metric system, 319-340
  - precision in, 313-314
  - and rates, 255-256
- Metric measure(s)
  - addition and subtraction of, 325-326
  - of capacity, 331-333
  - of length, 319-324
  - meaning of, 319
  - multiplication and division of, 326-328
  - of weight, 329-333
- Mixed numerals
  - addition with, 222
  - changing to fractional form, 194-195
  - multiplication and division with, 195,197,210
  - subtraction with, 237-240
- Multiple(s)
  - common, 180
  - least common, 181-183
  - meaning of, 179
- Multiplication
  - associative property of, 72-75, 172-173
  - closure property of, 67, 171
  - commutative property of, 68-71, 172-173
  - of decimals, 293-296
  - distributive property of, over addition, 76-82, 231-235
  - of dividend and divisor by same number, 207-208
  - estimating in, 293
  - identity element of, 69-71, 174-175
  - of measures, 326, 328
  - by multiples of ten, 83, 87; powers of ten,30
  - by one, 69-71, 174-175
  - of rational numbers, 169-173, 194-197
  - sets and, 64-66
  - of whole numbers, 64-82
  - by zero, 69-71

Number(s)
  common factors of, 158
  comparing, 184-187
  complete factorization of, 153-154
    156-157
  composite, 149-151
  factors of, 146
  greatest common factor of, 159-161
  powers of, 20-22
  prime, 152-157
  rational, 164-165
  reciprocal of, 204-206
  relatively prime, 160-161
  rounding, 283-288
Number line
  addition of integers on, 453-456
  addition and subtraction on, 215-218,
    236
  comparing numbers on, 189-191, 450-451
  graph of solution set, 269-272
  positive and negative integers on,
    448-452
  rational numbers on, 189-193
  in rounding numbers, 283-286
Numeral(s)
  base-five, 31-38
  base-ten, 16-26
  expanded, 17-19
  exponential form of, 20-22
  for fractional numbers, 166-168
  mixed, 194-197
  periods of, 24-26
  reading and writing of, 24-26
  Roman, 13-15
Numeration system
  base-five, 31-38
  decimal, 16-26
  Hindu-Arabic, 16
  Roman, 13-15
Numerator, 171, 175

Obtuse angle, 375-378
Octagon, 127
One
  division by, 69, 207, 212
  as a factor, 69-71
  names for, 174-175
  as identity element for multi-
    plication, 69-71, 174-175
Open sentence, 259-272
Operation with measures, 325-326, 328
Opposites, 445, 455
Order
  in the set of integers, 450-452
  in the set of rational numbers, 184-191

Parallel lines, 130-131
Parallelogram, 132
Pentagon, 127
Per cent
  as a decimal, 389-392
  as a fraction, 388-392
  meaning of, 386
  problems involving, 393-414
  as a ratio, 386
Perimeter
  of polygons, 341-346
  of rectangles and squares 343-346
Periods of a numeral, 24-26
Perpendicular lines, 375-378
Pi, 355
Place value
  in base-five numeration, 32-35
  in base-ten numeration, 16-26
  in decimals, 275-276
Plane(s)
  figure, 122
  intersection of, 116, 118
  lines and, 115-116, 118-119
  meaning of, 114
  points and, 115-119
Point(s)
  and lines, 105-108
  and line segments, 102-104
  naming a, 99
  and planes, 115-119
  sets of, 100
  and space, 98
Polygon
  kinds of, 126-129
  meaning, 125-126
Power(s) of a number, 20-22
Precision of measurement, 313-314
Prime factors, 152-153
Prime number
  meaning of, 148
  relatively prime, 160-161
Prism rectangular, 135
Problems
  analyzing, 61-62, 94-96
  with decimals, 292, 296, 301
  with per cent, 393-414
  with ratio, 253, 256
  with rational numbers, 173, 197, 202,
    212, 220-221, 226, 235, 239-240

  solving, 61-62, 94-96
  with whole numbers, 63, 96

Quadrilateral, 127
Quotient(s)
  estimating, 87-91
  expressing a rational number as, 165
  and ratios, 252

Radius, 347, 350
Rate
  of commission, 405-406
  of discount, 405-406
  of interest, 412-414
  of profit and loss, 408-410
  as a special kind of ratio, 255-256
Ratio
  expressing a, 242
  and fractions, 250-252
  meaning of, 241-242
  and measures, 254-256
  as an ordered number pair, 242
  rate as a special kind of, 255-256
  simplest form of a, 247-248, 252-253
Rational number(s)
  addition of, 215-227
  associative property of
    addition for, 224-227
    multiplication for, 172-173
  closure property of
    addition for, 223
    multiplication for, 171
  commutative property of
    addition for, 224-227
    multiplication for, 172-173
  comparing, 184-188
  distributive property for, 231-235
  division of, 207-212
  equivalent numerals for, 167
  meaning of, 165
  multiplication of, 169-171, 194-203
  multiplicative identity of, 174
  and number line, 189-191
  order in the set of, 184-188
  per cent names for, 386-392
  properties of, 172, 223, 224, 231
  renaming as decimals, 280-282
  set of, 165
  subtraction of, 236-240
Ray(s)
  and angles, 110-111
  endpoint of, 109-113
  meaning, 109
  naming, 110
  symbol, 110
Reciprocal, 204-207, 208-212
Rectangular prism, 135-136
Rectangular region
  area of, 360-363
  meaning of, 359
Region
  rectangular, 359
  square, 360
  square unit, 360
Regrouping
  in addition, 50
  of dividend, 85-86
Relatively prime numbers, 160-161
Remainder, 89-91
Replacement set, 263
Right angle, 374-378
Roman numeration system, 13-15
Rounding
  with decimals, 285-288
  with whole numbers, 283-284

Scale
  in bar graphs, 417
  in line graphs, 424-425
  in maps, 255-266
Segment, See *Line segment*
Sentence(s)
  equations, 260
  inequalities, 260
  and mathematical statements, 259
  open, 259-262
  solution sets of, 266-268
Set(s)
  and addition, 40-44
  disjoint, 40, 42-44
  and division, 93
  empty, 9-12
  finite, 7-8
  infinite, 8-9
  of integers, 442-444
  intersection of, 140-145
  members of, 3-4
  and multiplication, 64-66
  operations with, 41-44, 140-145
  of points, 100

  - product, 65-66
  - replacement, 263
  - and ratio, 241
  - of rational numbers, 165
  - and subsets, 10-12
  - and subtraction, 60
  - union of, 41-44
  - of whole numbers, 45
- Short form(s)
  - of addition and subtraction, 55-56
  - of division, 84-86
- Simple closed curve
  - boundary of, 122
  - interior of, 122
  - exterior of, 122
  - meaning, 121-122
- Solution of open sentences, 264-265
- Solution set
  - of equations, 266-268
  - graph of, 269-272
  - of inequalities, 266-268
- Solving problems, 59-63, 94-96
- Space figures, 134-138
- Sphere, 136-138
- Square
  - geometric figure, 126, 360
  - of a number, 20-22
- Square region, 360-361
- Standard numeral, 22, 25-26
- Standard unit of measure, 309, 360
- Statement, See Mathematical statement
- Subset(s), 10-12
- Subtraction
  - of decimals, 291-292
  - of measures, 325-326
  - problems involving, 61-63
  - of rational numbers, 236-240
  - of whole numbers, 60-61

- Ten
  - as a base, 27
  - multiplication by, 83
  - powers of, 20-21
- Trapezoid, 132
- Triangle, 127
- True statement, 259-260

- Union
  - of line segments, 125
  - of rays with common endpoint, 364
  - of sets, 41-44
- Unit(s)
  - angle, 365
  - line segment, 309-311
  - of linear measurement, 309-311
  - of square measure, 360-361

- Variable, 264
- Vertex, of an angle, 364

- Whole numbers
  - addition of, 50-58
  - associative property of
    - addition for, 50-58
    - multiplication for, 72-75
  - closure property of
    - addition for, 45
    - multiplication for, 67
  - commutative property of
    - addition for, 46-49
    - multiplication for, 68-71
  - distributive property of
    - multiplication over
    - addition for, 76-82
  - division of, 83-91
  - multiplication of, 64-82
  - problems involving, 92-96
  - subtraction of, 60-63

- Zero
  - as a factor, 69-71
  - as identity element for addition, 47-49
  - multiplicative property of, 69-71